Intrinsic Finite Element Methods for the Computation of Fluxes for Poisson's Equation

Philippe G. Ciarlet, Patrick Ciarlet, Stefan Sauter, C Simian

Intrinsic Finite Element Methods for the Computation of Fluxes for Poisson's Equation

P. G. Ciarlet* P. Ciarlet, Jr.† S. A. Sauter‡ C. Simian§

Abstract

In this paper we consider an intrinsic approach for the direct computation of the fluxes for problems in potential theory. We develop a general method for the derivation of intrinsic conforming and non-conforming finite element spaces and appropriate lifting operators for the evaluation of the right-hand side from abstract theoretical principles related to the second Strang Lemma. This intrinsic finite element method is analyzed and convergence with optimal order is proved.

2000 *Mathematics Subject Classification:* 65N30

Key words and phrases: elliptic boundary value problems, conforming and non-conforming finite element spaces, intrinsic formulation

The final publication is available at Springer via
http://dx.doi.org/10.1007/s00211-015-0730-9

1 Introduction

For the numerical solution of second order elliptic boundary value problems, Galerkin methods are nowadays among the most popular discretization methods. One can distinguish between the following types of Galerkin methods:

a) The continuous or exact variational formulation of the boundary value problem is employed and its discretization is achieved by replacing the infinite-dimensional energy space by either a finite dimensional *sub*space (conforming

*Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong (mapgc@cityu.edu.hk)

†Laboratoire POEMS, UMR 7231 CNRS/ENSTA/INRIA, ENSTA ParisTech, 828, boulevard des Maréchaux, 91762 Palaiseau Cedex, France (patrick.ciarlet@ensta-paristech.fr)

‡Institut für Mathematik, Universität Zürich, Winterthurerstr 190, CH-8057 Zürich, Switzerland (stas@math.uzh.ch)

§Department of Computer Science, University of Chicago, 1100 E 58th Street, Chicago, IL 60637, USA (csimian@cs.uchicago.edu)

Galerkin method) or by a finite dimensional space which is *not* a subspace of the energy space (non-conforming Galerking method). In the latter case, the volume or surface integrals involved in the continuous bilinear form are broken into a sum of local integrals. Standard examples for these finite dimensional spaces are conforming C^0 hp-finite elements, C^k spline spaces as they arise, e.g., in isogeometric analysis, and the Crouzeix-Raviart finite element.

b) The continuous variational formulation is modified by adding terms which enforce the continuity of the Galerkin solution in a weak way. This allows one to use discontinuous hp-finite element spaces without imposing any essential inter-element constraints in the definition of the spaces. The resulting methods are, e.g., non-conforming dG methods and non-conforming least squares methods.

Non-conforming Galerkin methods have nice properties, e.g. in different parts of the domain different discretizations can be easily used and glued together or, for certain classes of problems (Stokes problems, highly indefinite Helmholtz and Maxwell problems, problems with "locking", etc.), the non-conforming discretization enjoys a better stability behavior compared to the conforming one. But the computational cost is typically increased because additional integrals have to be evaluated on the element interfaces of the finite element mesh and, in addition, the total number of unknowns is increased compared to conforming methods. Moreover, the augmented discrete bilinear forms require certain mesh-depending control parameters whose choice for certain problem classes might be a delicate issue.

In this paper, our goal is two-fold: on the one hand, we will identify all piecewise polynomial finite element spaces which are *weakly non-conforming* in the sense that they are not contained in the continuous energy space but the (broken version of the) continuous bilinear form can still be used. In other words, we will address the question, how far can one go in the non-conforming direction while keeping the original forms?

On the other hand, we will develop a general method for the *derivation* of *intrinsic* conforming and non-conforming finite elements from theoretical principles for the discretization of elliptic partial differential equations. More precisely, we employ the stability and convergence theory for non-conforming finite elements based on the second Strang lemma and derive from these principles weak compatibility conditions for non-conforming finite elements. In the present case, we show that local polynomial finite element spaces for elliptic problems in divergence form *must* satisfy those compatibility conditions in order to be able to consistently estimate the perturbation term in the second Strang lemma.

As a simple model problem for the introduction of our method, we consider Poisson's equation but we emphasize that this method is applicable also for much more general (systems of) elliptic equations. We consider the intrinsic formulation of Poisson's equation, i.e., the minimization of the corresponding energy functional in the space of *admissible* energies as defined below. The goal is to construct element by element polynomial finite element spaces for the *direct* approximation of the physical quantity of interest, i.e., the flux, the electrostatic field, the velocity field, etc. depending on the underlying application. Furthermore, to take into account essential boundary conditions we have to construct a *lifting operator* as the left inverse of the elementwise gradient operator, that is, an operator defined element by element – whose realization turns out to be quite simple.

There is a vast literature on various conforming and non-conforming, primal, dual, mixed formulations of elliptic partial differential equations and conforming as well as non-conforming discretization. Our main focus is the development of a *concept* for deriving conforming and non-conforming intrinsic finite elements from theoretical principles and not the presentation of a specific new finite element space. For this reason, we do not provide an extensive list of references on the analysis of specific families of finite element spaces but refer to the monographs [6], [19], and [5], and the references therein.

Intrinsic formulations of the Lamé equations modelling linear three-dimensional elasticity have been first derived in [7]. An intrinsic finite element space has been developed in [8] and [9] by modifying the lowest order Nédélec finite elements (cf. [16], [17]) in such a way that the compatibility conditions which arise from the intrinsic formulation are exactly satisfied.

For Poisson's equation, the approach that we propose allows us to recover the non-conforming Crouzeix-Raviart element [12], the Fortin-Soulie element [13], the Crouzeix-Falk element [11], and the Gauss-Legendre elements [4], [21] as well as the standard conforming *hp*-finite elements.

The general theory of this paper will be developed for two-dimensional as well as for three-dimensional domains. However it turns out that the explicit construction of all non-conforming three-dimensional shape functions requires some further investigation of orthogonal polynomials on surfaces. So, we will essentially focus our attention on the two-dimensional case and present a single three-dimensional, non-conforming finite element at the end of the paper as an example.

The paper is organized as follows.

In Section 2 we introduce our model problem, Poisson's equation, and the relevant function spaces for the intrinsic formulation of the continuous problem as an energy minimization problem.

In Section 3 we derive weak continuity conditions for the characterization of the admissible energy space when the domain is split into simplices. Using these conditions, we derive conforming intrinsic polynomial finite element spaces and we show that they are (necessarily) the gradients of the well-known Lagrange *hp*-finite element spaces.

In Section 4 we focus on non-conforming discretizations. More precisely, we infer from the proof of the second Strang lemma appropriate compatibility conditions at the interfaces between elements of the mesh so that the non-conforming perturbation of the original bilinear form is consistent with the local error estimates. In two dimensions, we derive *all* types of piecewise polynomial finite elements that satisfy this condition and also derive local bases for these spaces. In three dimensions, we illustrate the construction by providing one example.

Finally, in Section 5 we summarize the main results and give some conclusions and some general comments on the construction of bases for the three-dimensional case.

2 Model problem

To formulate our model problem we first introduce some notation. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in $d = 2, 3$ dimensions. We denote by $\mathbf{e}^{(k)}$, $1 \leq k \leq d$, an orthonormal basis in $\mathbb{R}^d$, so that a point $\mathbf{x} \in \mathbb{R}^d$, can be expressed by its coordinates $(x_k)_{k=1}^d$ as $\mathbf{x} = \sum_{k=1}^d x_k \mathbf{e}^{(k)}$. The Euclidean scalar product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ is denoted by $\mathbf{a} \cdot \mathbf{b}$. To express the curl operator we introduce $d_* := 1$ if $d = 2$, and $d_* := 3$ if $d = 3$. The Euclidean scalar product in $\mathbb{R}^{d_*}$ is denoted, for $\mathtt{v}, \mathtt{w} \in \mathbb{R}^{d_*}$, by $\mathtt{v} \overset{*}{\cdot} \mathtt{w}$. The vector product $\times$ maps a pair of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ into $\mathbb{R}^{d_*}$ and is given by

$$\mathbf{a} \times \mathbf{b} := \begin{cases} a_1 b_2 - a_2 b_1 & \text{for } d = 2, \\ (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)^T & \text{for } d = 3. \end{cases}$$

The curl of a sufficiently smooth d-valued function $\mathbf{v}$ is equal to the d_*-valued function $\nabla \times \mathbf{v}$. The d-dimensional curl operator maps a sufficiently smooth d_*-valued function $\mathtt{v}$ to a d-valued function via

$$\mathbf{curl}\,(\mathtt{v}) := \begin{cases} \frac{\partial \mathtt{v}}{\partial x_2} \mathbf{e}^{(1)} - \frac{\partial \mathtt{v}}{\partial x_1} \mathbf{e}^{(2)}, & d = 2, \\ \\ \left(\frac{\partial \mathtt{v}_3}{\partial x_2} - \frac{\partial \mathtt{v}_2}{\partial x_3} \right) \mathbf{e}^{(1)} + \left(\frac{\partial \mathtt{v}_1}{\partial x_3} - \frac{\partial \mathtt{v}_3}{\partial x_1} \right) \mathbf{e}^{(2)} + \left(\frac{\partial \mathtt{v}_2}{\partial x_1} - \frac{\partial \mathtt{v}_1}{\partial x_2} \right) \mathbf{e}^{(3)}, & d = 3. \end{cases}$$

We consider the model problem of finding, for a given electric charge density $\rho \in H^{-1}(\Omega)$, an electrostatic field $\mathbf{e}$ in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which satisfies in a weak sense

$$-\operatorname{div}(\varepsilon \mathbf{e}) = \rho \qquad \text{in } \Omega, \tag{1}$$

where ε denotes the electrostatic permeability. In the electrostatic case, one may further write $\mathbf{e} = \nabla \phi$, where ϕ is the electrostatic potential, known up to a constant. We consider that the potential ϕ is constant on each connected component of the boundary $\partial\Omega$. This amounts to saying that (1) is complemented with a perfect conductor boundary condition, namely, $\gamma_\tau \mathbf{e} := (\mathbf{e} \times \mathbf{n})_{|\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outward normal vector field to $\partial\Omega$.

Throughout the paper we assume that

$$\Omega \subset \mathbb{R}^d \textit{ is a bounded Lipschitz domain with connected boundary } \partial\Omega. \tag{2}$$

As a consequence of this assumption, $\phi_{|\partial\Omega}$ is constant. Since ϕ is known up to a constant, we will assume without loss of generality that $\phi_{|\partial\Omega} = 0$.
Hence, the variational formulation of (1) restricted to the domain Ω is based on the space

$$\mathbf{E}(\Omega) := \nabla \left(H_0^1(\Omega) \right),$$

where $H_0^1(\Omega)$ denotes the usual Sobolev space and $\nabla \left(H_0^1(\Omega) \right)$ denotes its image under the gradient operator ∇.

Remark 1 *If $\partial\Omega$ consists of several disjoint connected components $\partial\Omega_k$, $0 \leq k \leq q$, where $q \geq 1$, i.e., $\partial\Omega = \bigcup_{k=0}^{q} \partial\Omega_k$, with $\overline{\partial\Omega_k} \cap \overline{\partial\Omega_{k'}} = \emptyset$ for $k \neq k'$, then*

$$\mathbf{E}(\Omega) = \left\{ \nabla v \mid \; v \in H^1(\Omega), \; v|_{\partial\Omega_0} = 0 \quad \text{and, for all } 1 \leq k \leq q, \quad v|_{\partial\Omega_k} = c_k \right\}$$

for arbitrary constants $c_k \in \mathbb{R}$, $1 \leq k \leq q$.

As a rule, we use boldface characters to denote functional spaces of d-valued functions, and typewriter characters to denote functional spaces of d_*-valued functions. Let $\mathbf{L}^2(\Omega) := \left(L^2(\Omega)\right)^d$, $\mathtt{H}^1(\Omega) := \left(H^1(\Omega)\right)^{d_*}$, $\mathtt{H}^{-1}(\Omega) := \left(\left(H_0^1(\Omega)\right)'\right)^{d_*}$, and $\mathtt{H}^{-1/2}(\partial\Omega) := \left(\left(H^{1/2}(\partial\Omega)\right)'\right)^{d_*}$. We recall a well-known result below, whose proof can be found in, e.g., [15].

Proposition 2 *Let $\Omega \subset \mathbb{R}^d$ satisfy (2). The operator $\nabla : H_0^1(\Omega) \to \mathbf{E}(\Omega)$ is an isomorphism and thus its inverse operator $\Lambda : \mathbf{E}(\Omega) \to H_0^1(\Omega)$ is continuous.*

It holds

$$\begin{aligned} \mathbf{E}(\Omega) &= \left\{ \mathbf{e} \in \mathbf{L}^2(\Omega) \mid \int_\Omega \mathbf{e} \cdot \mathbf{curl}(\mathtt{v}) = 0 \quad \forall \mathtt{v} \in \mathtt{H}^1(\Omega) \right\} \qquad (3) \\ &= \left\{ \mathbf{e} \in \mathbf{L}^2(\Omega) \mid \nabla \times \mathbf{e} = 0 \text{ in } \mathtt{H}^{-1}(\Omega) \;\textit{ and }\; \gamma_\tau \mathbf{e} = 0 \text{ in } \mathtt{H}^{-1/2}(\partial\Omega) \right\}. \end{aligned}$$

With the help of the inverse operator Λ, which we call a *lifting operator*, the variational formulation of the model problem reads: Find $\mathbf{e} \in \mathbf{E}(\Omega)$ such that

$$\int_\Omega \varepsilon \mathbf{e} \cdot \tilde{\mathbf{e}} = {}_{H^{-1}(\Omega)}\langle \rho, \Lambda\tilde{\mathbf{e}} \rangle_{H_0^1(\Omega)} \qquad \forall \tilde{\mathbf{e}} \in \mathbf{E}(\Omega), \tag{4}$$

where ${}_{H^{-1}(\Omega)}\langle \cdot, \cdot \rangle_{H_0^1(\Omega)}$ denotes the duality pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Under ad hoc assumptions on the permeability ε, e.g., $0 < \varepsilon_0 \leq \varepsilon(\mathbf{x}) \leq \varepsilon_1$ for almost all $\mathbf{x} \in \Omega$ for some constants ε_0 and ε_1, the solution $\mathbf{e}$ is the minimizer on $\mathbf{E}(\Omega)$ of the functional

$$j : \mathbf{E}(\Omega) \to \mathbb{R} \quad j(\tilde{\mathbf{e}}) := \frac{1}{2} \int_\Omega \varepsilon \tilde{\mathbf{e}} \cdot \tilde{\mathbf{e}} - {}_{H^{-1}(\Omega)}\langle \rho, \Lambda\tilde{\mathbf{e}} \rangle_{H_0^1(\Omega)}.$$

In most physical applications the quantity $\mathbf{e}$, or the flux $\varepsilon\mathbf{e}$, is the physical quantity of interest rather than the potential $u = \Lambda\mathbf{e}$. Hence, our goal is to *derive* conforming and non-conforming finite element spaces for the direct approximation of $\mathbf{e}$ in (4).

3 Conforming intrinsic finite element spaces

In this paper we restrict our studies to bounded, polygonal ($d = 2$) or polyhedral ($d = 3$) domains $\Omega \subset \mathbb{R}^d$ and geometrically conformal finite element meshes $\mathcal{T}$

[6] consisting of simplices τ. The local and global mesh width are denoted by $h_\tau := \operatorname{diam}\tau$ and $h := \max_{\tau\in\mathcal{T}} h_\tau$. The boundary of a simplex τ consists of $(d-1)$-dimensional simplices (facets for $d=3$ and triangle edges for $d=2$) which are denoted by F. We use in both cases the terminology “facet”. The set of all interior facets in $\mathcal{T}$ is denoted $\mathcal{F}$; the set of facets lying on $\partial\Omega$ is denoted $\mathcal{F}_{\partial\Omega}$. As a convention we assume that simplices and facets are closed sets. The interior of a simplex τ is denoted by $\overset{\circ}{\tau}$ and we write $\overset{\circ}{F}$ to denote the relative interior of a facet F. For a facet $F \in \mathcal{F}\cup\mathcal{F}_{\partial\Omega}$, let $\mathbf{n}_F$ denote a unit vector which is orthogonal to F. The orientation for the inner facets is arbitrary but fixed while the orientation for the boundary facets is such that $\mathbf{n}_F$ points toward the exterior of Ω.

For $p \in \mathbb{N}_0 := \{0,1,\ldots\}$, let $\mathbb{P}_d^p$ denote the space of d-variate polynomials of degree $\leq p$. For $\omega \subset \Omega$, let $\mathbb{P}_d^p(\omega)$ denote the restriction to ω of polynomials in $\mathbb{P}_d^p$. Given $\mathcal{T}$, we define the finite element spaces

$$\left.\begin{array}{l} S_{\mathcal{T}}^{p,m} := \left\{u \in H^{m+1}(\Omega) \mid \forall\tau\in\mathcal{T} : u|_{\overset{\circ}{\tau}} \in \mathbb{P}_d^p\right\}, \\ \mathbf{S}_{\mathcal{T}}^{p,m} := \left(S_{\mathcal{T}}^{p,m}\right)^d, \end{array}\right\} \quad \text{for } m=-1,0,$$

$$S_{\mathcal{T},0}^{p,0} := S_{\mathcal{T}}^{p,0} \cap H_0^1(\Omega),$$

and

$$\mathbf{E}_{\mathcal{T}}^p := \left\{\mathbf{e} \in \mathbf{S}_{\mathcal{T}}^{p,-1} \mid \int_\Omega \mathbf{e}\cdot\mathbf{curl}(\mathtt{v}) = 0 \quad \forall \mathtt{v} \in \mathtt{H}^1(\Omega)\right\}. \tag{5}$$

For $m=-1$, the spaces $S_{\mathcal{T}}^{p,-1}$, $\mathbf{S}_{\mathcal{T}}^{p,-1}$, $\mathbf{E}_{\mathcal{T}}^p$ consist of simplex-wise polynomials which are in general discontinuous across the facets. Hence the sum $u = \sum_i u_i$ of such functions is well defined in the interior of the simplices as well as the one-sided traces from the interior of a simplex towards its boundary.

For the inner facets $F \in \mathcal{F}$, we define the pointwise tangential jumps $[u]_F : F \to \mathbb{R}$ for $\mathbf{x} \in \overset{\circ}{F}$ by

$$[u]_F(\mathbf{x}) = \lim_{\varepsilon\searrow 0}\left(u(\mathbf{x}+\varepsilon\mathbf{n}_F) - u(\mathbf{x}-\varepsilon\mathbf{n}_F)\right). \tag{6}$$

We emphasize that the jump $[u]_F$ as the difference of the one-sided traces defines a continuous function on F. If the two one-sided limits at a facet F coincide we define u as this one-sided limit and thus u is well defined over F. If u is discontinuous across F, we avoid the definition of u on F and consider F as a set of measure zero. Note that the function u is continuous on $\overline{\Omega}$ if the jumps $[u]_F$ vanish for all inner facets.

From (3) we conclude that $\mathbf{E}_{\mathcal{T}}^p \subset \mathbf{E}(\Omega)$ is a piecewise polynomial finite element space which gives rise to the conforming Galerkin discretization of (4) by *intrinsic* finite elements: Find $\mathbf{e}_{\mathcal{T}} \in \mathbf{E}_{\mathcal{T}}^p$ such that

$$\int_\Omega \varepsilon\mathbf{e}_{\mathcal{T}}\cdot\tilde{\mathbf{e}}_{\mathcal{T}} = {}_{H^{-1}(\Omega)}\langle\rho,\Lambda\tilde{\mathbf{e}}\rangle_{H_0^1(\Omega)} \qquad \forall\tilde{\mathbf{e}}_{\mathcal{T}} \in \mathbf{E}_{\mathcal{T}}^p. \tag{7}$$

In the rest of Section 3, we will derive a local basis for $\mathbf{E}_{\mathcal{T}}^{p}$ and a realization of the lifting operator Λ. We define for later purpose the **piecewise gradient and curl operators** by

$$\nabla_{\mathcal{T}} u(\mathbf{x}) := \sum_{k=1}^{d} \frac{\partial u(\mathbf{x})}{\partial x_k} \mathbf{e}^{(k)}, \quad \nabla_{\mathcal{T}} \times \mathbf{e}(\mathbf{x}) := \nabla \times \mathbf{e}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega \backslash \left(\bigcup_{\tau \in \mathcal{T}} \partial \tau \right).$$

3.1 Local characterization of conforming intrinsic finite elements

In this section, we will develop a local characterization of conforming intrinsic finite elements. This approach generalizes that of [8], where such finite element approximations were considered for the first time (for the system of two-dimensional linearized elasticity).

Lemma 3 *The space $\mathbf{E}_{\mathcal{T}}^{p}$ can be characterized by local conditions according to*

$$\begin{aligned} \mathbf{E}_{\mathcal{T}}^{p} = \Big\{ \mathbf{e} \in \mathbf{S}_{\mathcal{T}}^{p,-1} \mid \nabla_{\mathcal{T}} \times \mathbf{e} = 0 \ , & \\ \text{and for all } F \in \mathcal{F} \quad & [\mathbf{e} \times \mathbf{n}_F]_F = 0 \ , \\ \text{and for all } F \in \mathcal{F}_{\partial\Omega} \quad & \mathbf{e} \times \mathbf{n}_F|_F = 0 \} \ . \end{aligned} \tag{8}$$

Proof. We denote the right-hand side in (8) by $\tilde{\mathbf{E}}_{\mathcal{T}}^{p}$ and prove that $\mathbf{E}_{\mathcal{T}}^{p} = \tilde{\mathbf{E}}_{\mathcal{T}}^{p}$. Let $\mathbf{e} \in \mathbf{E}_{\mathcal{T}}^{p}$. Consider the curl-condition (5) with test-fields $\mathtt{v}$.

Part a: For $\tau \in \mathcal{T}$, let $\mathtt{v} \in \mathtt{D}\left(\overset{\circ}{\tau}\right) := \left(\mathcal{D}\left(\overset{\circ}{\tau}\right)\right)^{d_*}$, where $\mathcal{D}\left(\overset{\circ}{\tau}\right) := C_c^{\infty}\left(\overset{\circ}{\tau}\right)$. Then,

$$\int_{\tau} (\nabla \times \mathbf{e}) \overset{*}{\cdot} \mathtt{v} = \int_{\tau} \mathbf{e} \cdot \mathbf{curl}(\mathtt{v}) = 0.$$

Since $\tau \in \mathcal{T}$ and $\mathtt{v} \in \mathtt{D}\left(\overset{\circ}{\tau}\right)$ are arbitrary, we conclude that $\nabla_{\mathcal{T}} \times \mathbf{e} = 0$ holds.

Part b: For $F \in \mathcal{F}$, let $\tau_1, \tau_2 \in \mathcal{T}$ be such that $F = \tau_1 \cap \tau_2$. We set $\omega_F := \tau_1 \cup \tau_2$. We choose $\mathtt{v} \in \mathtt{D}\left(\overset{\circ}{\omega}_F\right)$. Then

$$\int_{\tau_1} \mathbf{e} \cdot \mathbf{curl}(\mathtt{v}) + \int_{\tau_2} \mathbf{e} \cdot \mathbf{curl}(\mathtt{v}) = 0.$$

For $i = 1, 2$, denote by $\mathbf{n}^i$ the exterior normal for τ_i. Simplexwise integration by parts yields

$$\int_{\tau_i} \mathbf{e} \cdot \mathbf{curl}(\mathtt{v}) = \int_{\partial\tau_i} \left(\mathbf{e} \times \mathbf{n}^i\right) \overset{*}{\cdot} \mathtt{v} + \int_{\tau_i} (\nabla \times \mathbf{e}) \overset{*}{\cdot} \mathtt{v} \quad \text{for } d = 2, 3 \text{ and } i = 1, 2.$$

By adding the results for $i = 1, 2$ and taking into account $\mathtt{v} = 0$ on $\partial\omega_F$, we get

$$0 = \int_F \left(\mathbf{e} \times \mathbf{n}^1\right) \overset{*}{\cdot} \mathtt{v} + \int_F \left(\mathbf{e} \times \mathbf{n}^2\right) \overset{*}{\cdot} \mathtt{v} + \int_{\omega_F} (\nabla_{\mathcal{T}} \times \mathbf{e}) \overset{*}{\cdot} \mathtt{v}.$$

We already proved that $\nabla_{\mathcal{T}} \times \mathbf{e} = 0$, so that

$$0 = \int_F [\mathbf{e} \times \mathbf{n}_F]_F \overset{*}{\cdot} \mathtt{v}.$$

Since $\mathtt{v} \in \mathtt{D}\left(\overset{\circ}{\omega}_F\right)$ is arbitrary, we conclude $[\mathbf{e} \times \mathbf{n}_F]_F = 0$.

Part c: Let $F \in \mathcal{F}_{\partial\Omega}$ and $\tau \in \mathcal{T}$ such that $F \subset \partial\tau$. Let

$$\mathcal{D}_F(\tau) := \left\{ v|_\tau : v \in \mathcal{D}\left(\mathbb{R}^d\right) \text{ and } v = 0 \text{ in some neighborhood of } \Omega\backslash\tau \right\}.$$

Repeating the argument as in Part b by taking into account that $\mathtt{v} \in \mathtt{D}_F(\tau)$ in general does not vanish on F leads to $\mathbf{e} \times \mathbf{n}_F = 0$ in this case.

Thus, we have proved that $\mathbf{E}^p_{\mathcal{T}} \subset \tilde{\mathbf{E}}^p_{\mathcal{T}}$.

Part d: To prove the opposite inclusion we consider $\mathbf{e} \in \tilde{\mathbf{E}}^p_{\mathcal{T}}$. Then, for all $\mathtt{v} \in \mathtt{H}^1(\Omega)$ it holds by integration by parts

$$\begin{aligned}
\int_\Omega \mathbf{e} \cdot \mathbf{curl}(\mathtt{v}) &= \sum_{\tau \in \mathcal{T}} \int_\tau \mathbf{e} \cdot \mathbf{curl}(\mathtt{v}) \\
&= \sum_{\tau \in \mathcal{T}} \int_\tau (\nabla_{\mathcal{T}} \times \mathbf{e}) \overset{*}{\cdot} \mathtt{v} + \sum_{\tau \in \mathcal{T}} \int_{\partial\tau} (\mathbf{e} \times \mathbf{n}^\tau) \overset{*}{\cdot} \mathtt{v} \\
&= \sum_{\tau \in \mathcal{T}} \int_\tau (\nabla_{\mathcal{T}} \times \mathbf{e}) \overset{*}{\cdot} \mathtt{v} + \sum_{F \in \mathcal{F}} \int_F s_F [\mathbf{e} \times \mathbf{n}_F]_F \overset{*}{\cdot} \mathtt{v} \\
&\qquad + \sum_{F \in \mathcal{F}_{\partial\Omega}} \int_F (\mathbf{e} \times \mathbf{n}_F) \overset{*}{\cdot} \mathtt{v} \\
&= 0.
\end{aligned}$$

Above, $s_F = \pm 1$ depending on the orientation of the facet F. Hence, $\tilde{\mathbf{E}}^p_{\mathcal{T}} \subset \mathbf{E}^p_{\mathcal{T}}$ and the assertion follows. ∎

3.2 Integration

We start with a lemma on integration of curl-free polynomials. Let

$$\mathbf{P}^p_{\text{curl}} := \left\{ \mathbf{e} \in (\mathbb{P}^p_d)^d \mid \nabla \times \mathbf{e} = 0 \right\} \tag{9}$$

and, for $\tau \in \mathcal{T}$, let $\mathbf{P}^p_{\text{curl}}(\tau) := \{ \mathbf{e}|_\tau : \mathbf{e} \in \mathbf{P}^p_{\text{curl}} \}$.

Lemma 4 *For any $\tau \in \mathcal{T}$ and any $\mathbf{e} \in \mathbf{P}^p_{\text{curl}}(\tau)$, it holds*

$$\emptyset \neq \left\{ u \in H^1(\tau) \mid \nabla u = \mathbf{e} \right\} \subset \mathbb{P}^{p+1}_d(\tau). \tag{10}$$

Proof. Let $\tau \in \mathcal{T}$ and $\mathbf{e} \in \mathbf{P}^p_{\text{curl}}(\tau)$. In [15, 2] it is proved that there exists $u \in H^1(\tau)$, unique up to a constant, such that $\nabla u = \mathbf{e}$; hence the left-hand side in (10) is proved. Let $\mathbf{m}_\tau$ be the center of mass for τ. Then Poincaré's theorem yields that the path integral

$$U(\mathbf{x}) := \int_{\gamma_{\mathbf{x}}} \mathbf{e} \quad \text{with} \quad \gamma_{\mathbf{x}} \text{ denoting the straight path } \overline{\mathbf{m}_\tau \mathbf{x}} \tag{11}$$

defines $U \in H^1(\tau)$ such that $\nabla U = \mathbf{e}$. Since $\mathbf{e} \in \mathbf{P}^p_{\text{curl}}(\tau)$, there are coefficients $\mathbf{a}_\mu \in \mathbb{R}^d$ such that

$$\mathbf{e}(\mathbf{x}) = \sum_{|\mu| \leq p} \mathbf{a}_\mu (\mathbf{x} - \mathbf{m}_\tau)^\mu$$

with the usual multi-index notation $\mu \in \mathbb{N}_0^d$, $|\mu| := \mu_1 + \ldots + \mu_d$, $\mathbf{w}^\mu := w_1^{\mu_1} \cdots w_d^{\mu_d}$. To evaluate the integral in (11) we employ the affine pullback $\chi_{\mathbf{x}} : [0,1] \to \overline{\mathbf{m}_\tau \mathbf{x}}$, $\chi_{\mathbf{x}} := \mathbf{m}_\tau + t(\mathbf{x} - \mathbf{m}_\tau)$ and obtain

$$\begin{aligned}
U(\mathbf{x}) &= \int_0^1 \mathbf{e} \circ \chi_{\mathbf{x}}(t) \cdot \chi'_{\mathbf{x}}(t)\, dt \\
&= \sum_{|\mu| \leq p} \mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau) \int_0^1 (t(\mathbf{x} - \mathbf{m}_\tau))^\mu\, dt \\
&= \sum_{|\mu| \leq p} (\mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau))(\mathbf{x} - \mathbf{m}_\tau)^\mu \int_0^1 t^{|\mu|} dt \\
&= \sum_{|\mu| \leq p} \mathbf{a}_\mu \cdot (\mathbf{x} - \mathbf{m}_\tau) \frac{(\mathbf{x} - \mathbf{m}_\tau)^\mu}{|\mu| + 1} \in \mathbb{P}_d^{p+1}.
\end{aligned}$$

Since the functions in the set $\left\{u \in H^1(\tau) \mid \nabla u = \mathbf{e}\right\}$ in (10) differ only by a constant we have proved the second inclusion in (10). ■

Lemma 4 motivates the definition of the local lifting operator $\lambda_\tau^c : \mathbf{P}^p_{\text{curl}}(\tau) \to \mathbb{P}_d^{p+1}(\tau)$ with $\tau \in \mathcal{T}$, $c \in \mathbb{R}$ given, for $\mathbf{e} \in \mathbf{P}^p_{\text{curl}}(\tau)$, by

$$\lambda_\tau^c(\mathbf{e}) := U + c \qquad \text{with } U \text{ as in (11).} \tag{12}$$

Note that the space in (10) satisfies

$$\left\{u \in H^1(\tau) \mid \nabla u = \mathbf{e}\right\} = \left\{\lambda_\tau^c(\mathbf{e}) : c \in \mathbb{R}\right\}.$$

Corollary 5 *The (restriction of the) operator $\Lambda : \mathbf{E}^p_{\mathcal{T}} \to S^{p+1,0}_{\mathcal{T},0}$ is an isomorphism with inverse $\nabla : S^{p+1,0}_{\mathcal{T},0} \to \mathbf{E}^p_{\mathcal{T}}$.*

Proof. From Lemma 4 we conclude that

$$\Lambda \mathbf{E}^p_{\mathcal{T}} \subset S^{p+1,-1}_{\mathcal{T}}$$

holds. Since $\mathbf{E}^p_{\mathcal{T}} \subset \mathbf{E}$, the properties of the lifting operator Λ imply that

$$\Lambda \mathbf{E}^p_{\mathcal{T}} \subset H^1_0(\Omega).$$

Hence

$$\Lambda \mathbf{E}_{\mathcal{T}}^{p} \subset S_{\mathcal{T}}^{p+1,-1} \cap H_0^1(\Omega) = S_{\mathcal{T},0}^{p+1,0}.$$

On the other hand, we have $S_{\mathcal{T},0}^{p+1,0} \subset H_0^1(\Omega)$ and hence $\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{E}$. Furthermore, it is clear that

$$\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{S}_{\mathcal{T}}^{p,-1}.$$

Hence

$$\nabla S_{\mathcal{T},0}^{p+1,0} \subset \mathbf{S}_{\mathcal{T}}^{p,-1} \cap \mathbf{E} = \mathbf{E}_{\mathcal{T}}^{p}$$

from which we finally conclude that the inclusion

$$S_{\mathcal{T},0}^{p+1,0} \subset \Lambda \mathbf{E}_{\mathcal{T}}^{p}$$

holds. ∎

3.3 A Local basis for conforming intrinsic finite elements

Corollary 5 shows that a local basis for $\mathbf{E}_{\mathcal{T}}^{p}$ can be easily constructed by using the standard basis functions for *hp*-finite element spaces (cf. [19]). We recall briefly their definition. Let

$$\widehat{\mathcal{N}}^p := \left\{ \frac{\mathbf{i}}{p} : \mathbf{i} \in \mathbb{N}_0^d \text{ with } i_1 + \ldots + i_d \leq p \right\}$$

denote the unisolvent set of equi-spaced nodal points on the d-dimensional unit simplex

$$\hat{\tau}_d := \left\{ \mathbf{x} \in \mathbb{R}_{\geq 0}^d \mid x_1 + \ldots + x_d \leq 1 \right\}. \tag{13}$$

For a simplex $\tau \in \mathcal{T}$ with vertices $\mathbf{A}_i^\tau$, $0 \leq i \leq d$, let $\chi_\tau : \hat{\tau}_d \to \tau$ denote the affine mapping $\chi_\tau(\hat{\mathbf{x}}) := \mathbf{A}_0^\tau + \sum_{i=1}^d (\mathbf{A}_i^\tau - \mathbf{A}_0^\tau)\,\hat{x}_i$. Then the set of interior nodal points are given by

$$\mathcal{N}^p := \left\{ \chi_\tau\left(\hat{N}\right) \mid \hat{N} \in \widehat{\mathcal{N}}^p, \tau \in \mathcal{T} \right\} \backslash \partial\Omega. \tag{14}$$

The Lagrange basis for $S_{\mathcal{T},0}^{p,0}$ can be indexed by the nodal points $N \in \mathcal{N}^p$ and is characterized by

$$b_{p,N}^{\mathcal{T}} \in S_{\mathcal{T},0}^{p,0} \quad \text{and} \quad \forall N' \in \mathcal{N}^p \quad b_{p,N}^{\mathcal{T}}(N') = \begin{cases} 1 & N = N', \\ 0 & N \neq N'. \end{cases} \tag{15}$$

Recall that the simplices in $\mathcal{T}$ are by convention closed sets and the facets in $\mathcal{F} \cup \mathcal{F}_{\partial\Omega}$ are closed as well. Let $\mathcal{V}$ (respectively $\mathcal{V}_{\partial\Omega}$) denote the inner vertices (resp. boundary vertices) of the mesh $\mathcal{T}$. For $d = 3$, we let $\mathcal{E}$ denote the set of all interior $(d-2)$-dimensional closed simplex edges, that is, all those edges that are not subsets of $\partial\Omega$.

Definition 6 *For all $\tau \in \mathcal{T}$, $F \in \mathcal{F}$, $E \in \mathcal{E}$ and for $d = 3$, $V \in \mathcal{V}$, the spaces $\mathbf{B}_\tau^p$, $\mathbf{B}_F^p$, $\mathbf{B}_E^p$ and for $d = 3$, the space $\mathbf{B}_V^p$ are given as the following spans of basis functions:*

$$\begin{aligned}
\mathbf{B}_\tau^p &:= \operatorname{span}\left\{\nabla b_{p+1,N}^{\mathcal{T}} \mid N \in \overset{\circ}{\tau} \cap \mathcal{N}^{p+1}\right\}, \\
\mathbf{B}_F^p &:= \operatorname{span}\left\{\nabla b_{p+1,N}^{\mathcal{T}} \mid N \in \overset{\circ}{F} \cap \mathcal{N}^{p+1}\right\}, \\
\mathbf{B}_E^p &:= \operatorname{span}\left\{\nabla b_{p+1,N}^{\mathcal{T}} \mid N \in \overset{\circ}{E} \cap \mathcal{N}^{p+1}\right\} \text{ (for } d = 3\text{)}, \\
\mathbf{B}_V^p &:= \operatorname{span}\left\{\nabla b_{p+1,V}^{\mathcal{T}}\right\}.
\end{aligned}$$

The following proposition shows that these spaces give rise to a direct sum decomposition and that these spaces are locally defined. To be more specific, we first have to introduce some notation.

For any facet $F \in \mathcal{F}$, vertex $V \in \mathcal{V}$, and $E \in \mathcal{E}$ we define the sets

$$\begin{aligned}
\mathcal{T}_F &:= \{\tau \in \mathcal{T} : F \subset \partial\tau\}, & \omega_F &:= \bigcup_{\tau \in \mathcal{T}_F} \tau, \\
\mathcal{T}_V &:= \{\tau \in \mathcal{T} : V \in \tau\}, & \omega_V &:= \bigcup_{\tau \in \mathcal{T}_V} \tau, \\
\mathcal{T}_E &:= \{\tau \in \mathcal{T} : E \subset \tau\}, & \omega_E &:= \bigcup_{\tau \in \mathcal{T}_E} \tau \quad \text{for } d = 3, \\
\mathcal{F}_V &:= \{F \in \mathcal{F} : V \in \partial F\}, & & \text{for } d = 2.
\end{aligned} \tag{16}$$

Proposition 7 *Let $\mathbf{B}_\tau^p$, $\mathbf{B}_F^p$, $\mathbf{B}_E^p$, $\mathbf{B}_V^p$ be as in Definition 6. Then the following direct sum decomposition holds:*

$$\mathbf{E}_{\mathcal{T}}^p = \begin{cases} \left(\bigoplus_{V \in \mathcal{V}} \mathbf{B}_V^p\right) \oplus \left(\bigoplus_{F \in \mathcal{F}} \mathbf{B}_F^p\right) \oplus \left(\bigoplus_{\tau \in \mathcal{T}} \mathbf{B}_\tau^p\right) & d = 2, \\ \left(\bigoplus_{V \in \mathcal{V}} \mathbf{B}_V^p\right) \oplus \left(\bigoplus_{E \in \mathcal{E}} \mathbf{B}_E^p\right) \oplus \left(\bigoplus_{F \in \mathcal{F}} \mathbf{B}_F^p\right) \oplus \left(\bigoplus_{\tau \in \mathcal{T}} \mathbf{B}_\tau^p\right) & d = 3. \end{cases} \tag{17}$$

For any simplex τ, one can further identify $\mathbf{B}_\tau^p$ with the subspace of elements of $\mathbf{E}_{\mathcal{T}}^p$ supported in τ, namely:

$$\mathbf{B}_\tau^p := \left\{\mathbf{e} \in \mathbf{E}_{\mathcal{T}}^p \mid \operatorname{supp} \mathbf{e} \subset \tau\right\}. \tag{18}$$

For any facet $F \in \mathcal{F}$ and $\mathbf{e} \in \mathbf{B}_F^p$, it holds

$$\operatorname{supp} \mathbf{e} \subset \omega_F. \tag{19}$$

For any vertex $V \in \mathcal{V}$ and $\mathbf{e} \in \mathbf{B}_V^p$, it holds

$$\operatorname{supp} \mathbf{e}_V \subset \omega_V. \tag{20}$$

Let $d = 3$. For any edge $E \in \mathcal{E}$ and $\mathbf{e} \in \mathbf{B}_E^p$, it holds

$$\operatorname{supp} \mathbf{e} \subset \omega_E.$$

Proof. Corollary 5 implies that $(\nabla b^{\mathcal{T}}_{p+1,N})_{N\in\mathcal{N}^{p+1}}$ is a basis of $\mathbf{E}^{p}_{\mathcal{T}}$. The assertion follows simply by sorting these basis functions, according as to whether they are associated with a single simplex, with two simplices with a facet in common, with simplices with a vertex in common, and for $d=3$ with simplices with an edge in common.

The properties for the local supports are direct consequences of the corresponding properties of standard nodal basis as defined in (15). ∎

Remark 8 *Proposition 7 shows that the intrinsic finite element formulation (7) is equivalent to the standard Galerkin finite element formulation of (1): Find* $u_{\mathcal{T}} \in S^{p+1,0}_{\mathcal{T},0}$ such that

$$\int_{\Omega} \varepsilon \nabla u_{\mathcal{T}} \cdot \nabla v_{\mathcal{T}} = {}_{H^{-1}(\Omega)}\langle \rho, v_{\mathcal{T}} \rangle_{H^1_0(\Omega)} \qquad \forall v_{\mathcal{T}} \in S^{p+1,0}_{\mathcal{T},0}$$

with $\mathbf{e}_{\mathcal{T}} = \nabla u_{\mathcal{T}}$. *However, the derivation via the intrinsic variational formulation has the advantage of providing insights on how to design non-conforming intrinsic finite elements.*

4 Non-conforming intrinsic finite elements

In order to ensure existence and uniqueness of the solution to the variational formulation and to obtain convergence estimates for the finite element discretization we impose from now on that $\rho \in L^2(\Omega)$, so that we may replace duality products by integrals, and we make the following assumptions on the electrostatic permeability: The electrostatic permeability ε in (1) satisfies $\varepsilon \in L^{\infty}(\Omega)$ and

$$0 < \varepsilon_{\min} := \operatorname*{ess\,inf}_{\mathbf{x}\in\Omega} \varepsilon(\mathbf{x}) \leq \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} \varepsilon(\mathbf{x}) =: \varepsilon_{\max} < \infty. \tag{21}$$

Besides, there exists a partition $\mathcal{P} := (\Omega_j)_{j=1}^{J}$ of Ω into J polygons (polyhedra for $d=3$) such that, for some $r \geq 1$,

$$\|\varepsilon\|_{PW^{r,\infty}(\Omega)} := \max_{1\leq j\leq J} \left\| \varepsilon|_{\Omega_j} \right\|_{W^{r,\infty}(\Omega_j)} < \infty. \tag{22}$$

Remark 9 *In practical situations, one may have to deal with a partition into curved polygons or polyhedra, of a domain with piecewise curved boundary. In this case one should consider isoparametric finite elements. For simplicity, we restrict ourselves to the case of affine finite elements, and hence to piecewise polygons or polyhedra.*

4.1 Definition of non-conforming intrinsic finite elements

In this section, we will define non-conforming intrinsic finite element spaces in order to approximate the solution of (4). As a minimal requirement we assume that the non-conforming finite element space $\mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}}$ satisfies

$$\mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}} \subset \mathbf{L}^2(\Omega) \quad \text{and} \quad \mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}} \not\subset \mathbf{E}(\Omega) \quad \text{and} \quad \dim \mathbf{E}^{p}_{\mathcal{T},\mathrm{nc}} < \infty. \tag{23}$$

We further require that $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ is a piecewise polynomial, simplex by simplex curl-free finite element space and that the conforming space $\mathbf{E}^p_{\mathcal{T}}$ is a subspace of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$:

$$\mathbf{E}^p_{\mathcal{T}} \subset \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \subset \left\{ \mathbf{e} \in \mathbf{S}^{p,-1}_{\mathcal{T}} \mid \nabla_{\mathcal{T}\times\mathbf{e}} = 0 \right\}. \tag{24}$$

To be able to define a variational formulation in $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$, we have to extend the lifting operator Λ to an operator $\Lambda_{\mathcal{T}}$ whose image satisfies the following properties

$$\Lambda_{\mathcal{T}} : \left(\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} + \mathbf{E}(\Omega) \right) \to L^2(\Omega) \tag{25}$$

$$\Lambda_{\mathcal{T}} : \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \to S^{p+1,-1}_{\mathcal{T}} \tag{26}$$

as well as the consistency condition

$$\Lambda_{\mathcal{T}}\mathbf{e} = \Lambda\mathbf{e} \qquad \forall \mathbf{e} \in \mathbf{E}(\Omega). \tag{27}$$

The complete definitions of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ and $\Lambda_{\mathcal{T}}$ will be based on the convergence theory for non-conforming finite elements according to the second Strang lemma (cf. [6, Th. 4.2.2]): this will tell us how to define them and obtain in the end an optimal order of convergence (see Theorem 15 hereafter).

In the same spirit as in Section 3, we first define the operator $\Lambda_{\mathcal{T}}$ simplexwise by the local lifting operators λ^c_τ as in (12):

$$\left(\Lambda_{\mathcal{T}}\mathbf{e}\right)\big|_{\overset{\circ}{\tau}} := \lambda^{c_\tau}_\tau \left(\mathbf{e}|_{\overset{\circ}{\tau}} \right) \in \mathbb{P}^{p+1}_d \left(\overset{\circ}{\tau} \right) \qquad \forall \tau \in \mathcal{T} \quad \forall \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}. \tag{28}$$

Note that the coefficients $(c_\tau)_{\tau\in\mathcal{T}}$ are at our disposal.

From (28) we conclude that $\nabla_{\mathcal{T}}$ is a left-inverse to $\Lambda_{\mathcal{T}}$, i.e.,

$$\forall \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} : \nabla_{\mathcal{T}}\Lambda_{\mathcal{T}}\mathbf{e} = \mathbf{e}. \tag{29}$$

A compatibility assumption on $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ concerning the jumps of functions across facets is formulated next. For a facet F with vertices $\mathbf{A}^F_i$, $0 \leq i \leq d-1$, the affine mapping $\chi_F : \hat{\tau}_{d-1} \to F$ (with $\widehat{\tau}_{d-1}$ as in (13)) is given by $\chi_F(\xi) = \mathbf{A}^F_0 + \sum_{i=1}^{d-1} \left(\mathbf{A}^F_i - \mathbf{A}^F_0 \right) \xi_i$. The space of $(d-1)$-variate polynomials of degree $\leq p$ on F is given by

$$\mathbb{P}^p_{d-1}(F) := \left\{ q \circ \chi_F^{-1} \mid q \text{ is a polynomial of degree } \leq p \text{ on } \hat{\tau}_{d-1} \right\}. \tag{30}$$

On the one hand, given $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T}}$, one has $[\Lambda_{\mathcal{T}}\mathbf{e}]_F = 0$ for all $F \in \mathcal{F}$, and $\Lambda_{\mathcal{T}}\mathbf{e} = 0$ on $\partial\Omega$. On the other hand, for elements of the non-conforming finite element space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$, we require that these conditions are *weakly* enforced. Given $\tilde{\mathbf{e}} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$, keeping in mind that, along every facet $F \in \mathcal{F}$ (respectively $F \in \mathcal{F}_{\partial\Omega}$), the jump $[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_F$ (resp. the value $\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}$) is a polynomial of degree $\leq (p+1)$, we choose a *weak facet compatibility condition* that reads:

$$\begin{aligned} &\int_F [\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_F \, q = 0 \quad \forall q \in \mathbb{P}^p_{d-1}(F), \ \forall F \in \mathcal{F} \quad \text{and} \\ &\int_F \Lambda_{\mathcal{T}}\tilde{\mathbf{e}} \, q = 0 \quad \forall q \in \mathbb{P}^p_{d-1}(F), \ \forall F \in \mathcal{F}_{\partial\Omega}. \end{aligned} \tag{31}$$

Remark 10 *One has the freedom to choose a priori the degree of the polynomials q between 0 and $p+1$ so that the interelement continuity can be weakened in a flexible way. Indeed, a degree equal to $p+1$ defines conforming finite elements, because (31) then implies $[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_F = 0$ across all interior facets F, and $\Lambda_{\mathcal{T}}\tilde{\mathbf{e}} = 0$ on $\partial\Omega$, and Lemma 3 leads to $\tilde{\mathbf{e}} \in \mathbf{E}^p_{\mathcal{T}}$. On the other hand, a degree strictly lower than $p+1$ in the implicit definition (31) of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ leads to a non-conforming finite element space, such that $\mathbf{E}^p_{\mathcal{T}}$ is a strict subset of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$. The degree of the polynomials q, which is chosen here equal to p, actually yields an optimal order of convergence (see Theorem 15), whereas a degree strictly lower than p yields a sub-optimal order of convergence.*

These considerations are summarized in the following definition.

Definition 11 *The non-conforming intrinsic finite element space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ is given by*

$$\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} := \left\{ \mathbf{e} \in \mathbf{S}^{p,-1}_{\mathcal{T}} \mid \nabla_{\mathcal{T}} \times \mathbf{e} = 0 \quad \textit{and} \quad \textit{(31) is satisfied} \right\}.$$

This definition directly implies that condition (24), i.e., $\mathbf{E}^p_{\mathcal{T}} \subset \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ holds.

In Section 4.2 we will prove for the two-dimensional case the following direct sum decomposition

$$\begin{aligned} \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} = & \mathbf{E}^p_{\mathcal{T}} \oplus \bigoplus_{F \in \mathcal{F}} \operatorname{span} \left\{ \nabla_{\mathcal{T}} U^F_{p+1,k} : 1 \le k \le N_{\mathrm{facet}} \right\} \\ & \oplus \bigoplus_{\tau \in \mathcal{T}} \operatorname{span} \left\{ \nabla_{\mathcal{T}} U^\tau_{p+1,k} : 1 \le k \le N_{\mathrm{simplex}} \right\}, \\ & \text{with } \operatorname{supp} U^\tau_{p+1,k} \subset \tau \text{ and } \operatorname{supp} U^F_{p+1,k} \subset \omega_F \end{aligned} \tag{32}$$

for some non-conforming functions $U^F_{p+1,k}$ and $U^\tau_{p+1,k}$ which will be defined in Section 4.2. The numbers N_{facet}, N_{simplex} both depend on the dimension d and on the degree of approximation p.

Remark 12 *For $d = 2$, we have $N_{\mathrm{facet}} = 1$ and $N_{\mathrm{simplex}} = 0$ for even p, i.e., only (one) facet-oriented, non-conforming basis function arises, while for odd p it holds that, vice versa, $N_{\mathrm{facet}} = 0$ and $N_{\mathrm{simplex}} = 1$, i.e., there is only (one) simplex-oriented, non-conforming basis function. The functions $U^F_{p+1} := U^F_{p+1,k}$ and $U^\tau_{p+1} := U^\tau_{p+1,k}$ will be respectively defined in(45) and (49). The case $d = 3$ will be considered in the forthcoming paper [10].*

As a consequence of (32), one deduces the following definition of the *extended lifting operator*.

Definition 13 *For a function $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ written as*

$$\mathbf{e} = \mathbf{e}_1 + \sum_{F \in \mathcal{F}} \sum_{k=1}^{N_{\mathrm{facet}}} \alpha_{F,k} \nabla_{\mathcal{T}} U^F_{p+1,k} + \sum_{\tau \in \mathcal{T}} \sum_{k=1}^{N_{\mathrm{simplex}}} \alpha_{\tau,k} \nabla_{\mathcal{T}} U^\tau_{p+1,k} \tag{33}$$

for some $\mathbf{e}_1 \in \mathbf{E}_{\mathcal{T}}^{p}$ *and coefficients* $\alpha_{F,k}$ *resp.* $\alpha_{\tau,k}$, *the extended lifting operator* $\Lambda_{\mathcal{T}}$ *is defined by*

$$\Lambda_{\mathcal{T}}\mathbf{e} := \Lambda\mathbf{e}_1 + \sum_{F\in\mathcal{F}}\sum_{k=1}^{N_{\text{facet}}} \alpha_{F,k} U_{p+1,k}^{F} + \sum_{\tau\in\mathcal{T}}\sum_{k=1}^{N_{\text{simplex}}} \alpha_{\tau,k} U_{p+1,k}^{\tau}.$$

We now prove an important result on the locality of the lifting operator $\Lambda_{\mathcal{T}}$.

Proposition 14 *Assume that (32) holds. For any* $\mathbf{e} \in \mathbf{E}_{\mathcal{T},\text{nc}}^{p}$ *with connected support* $\omega_{\mathbf{e}}$ *which fulfills the condition that for all disjoint connected components* $(\omega_j)_j$ *of* $\Omega\backslash\omega_{\mathbf{e}}$, $\overline{\omega_j} \cap \partial\Omega$ *has positive boundary measure, it holds*

$$\operatorname{supp}\Lambda_{\mathcal{T}}\mathbf{e} \subset \omega_{\mathbf{e}}.$$

Proof. We split $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$ according to (33) with $\mathbf{e}_1 \in \mathbf{E}$. Since the sum, in (32), is direct we conclude[1] that $\operatorname{supp}\mathbf{e}_i \subset \omega_{\mathbf{e}}$ for $i = 1, 2$. From Proposition 2 we obtain $\Lambda_{\mathcal{T}}\mathbf{e}_1 = \Lambda\mathbf{e}_1 \in H_0^1(\Omega)$. Since $\mathbf{e}_1|_{\Omega\backslash\omega_{\mathbf{e}}} = 0$ Poincaré's theorem implies that $\Lambda\mathbf{e}_1|_{\omega_j} = c_j$, i.e., $\Lambda\mathbf{e}_1$ is constant on each disjoint connected component ω_j of $\Omega\backslash\omega_{\mathbf{e}}$. Since $\overline{\omega_j} \cap \partial\Omega$ has positive boundary measure, the property $\Lambda\mathbf{e}_1 \in H_0^1(\Omega)$ implies that $\Lambda\mathbf{e}_1|_{\omega_j} = 0$. This proves $\operatorname{supp}\Lambda_{\mathcal{T}}\mathbf{e}_1 \subset \omega_{\mathbf{e}}$.

According to the definition of $\Lambda_{\mathcal{T}}$ for the non-conforming part $\mathbf{e}_2$, which implies in particular that $\Lambda_{\mathcal{T}}\left(\nabla_{\mathcal{T}}U_{p+1,k}^{F}\right) = U_{p+1,k}^{F}$, one gets that $\operatorname{supp}\nabla_{\mathcal{T}}U_{p+1,k}^{F} = \operatorname{supp}U_{p+1,k}^{F}$ so that $\operatorname{supp}\Lambda_{\mathcal{T}}\mathbf{e}_2 \subset \omega_{\mathbf{e}}$. The proof for the functions $U_{p+1,k}^{\tau}$ is by an analogous argument. ∎

Note that, for any inner facet $F \in \mathcal{F}$, we may choose $q = 1$ in the left condition of (31) to obtain $\int_F [\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_F = 0$: hence, the jump $[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_F$ is always zero-mean valued. Let h_F denote the diameter of F. The combination of a Poincaré inequality with a trace inequality then yields

$$\begin{aligned} \left\|[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_F\right\|_{L^2(F)} &\leq Ch_F \left\|[\nabla_{\mathcal{T}}\Lambda_{\mathcal{T}}\tilde{\mathbf{e}} \times \mathbf{n}_F]_F\right\|_{L^2(F)} \\ &\overset{(29)}{=} Ch_F \left\|[\tilde{\mathbf{e}} \times \mathbf{n}_F]_F\right\|_{L^2(F)} \leq \tilde{C}h_F^{1/2} \left\|\tilde{\mathbf{e}}\right\|_{L^2(\omega_F)}, \end{aligned} \tag{34}$$

for some constants C and $\tilde{C}$. In a similar fashion we obtain for all boundary facets $F \in \mathcal{F}_{\partial\Omega}$ and all $\mathbf{e} \in \mathbf{E}_{\mathcal{T},\text{nc}}^{p}$ the estimate

$$\left\|\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}\right\|_{L^2(F)} \leq \tilde{C}h_F^{1/2} \left\|\tilde{\mathbf{e}}\right\|_{L^2(\omega_F)}. \tag{35}$$

Equipped with $\mathbf{E}_{\mathcal{T},\text{nc}}^{p}$ and $\Lambda_{\mathcal{T}}$, the non-conforming Galerkin discretization of (4) reads: Find $\mathbf{e}_{\mathcal{T}} \in \mathbf{E}_{\mathcal{T},\text{nc}}^{p}$ such that

$$\int_\Omega \varepsilon\mathbf{e}_{\mathcal{T}} \cdot \tilde{\mathbf{e}} = \int_\Omega \rho\Lambda_{\mathcal{T}}\tilde{\mathbf{e}} \qquad \forall\tilde{\mathbf{e}} \in \mathbf{E}_{\mathcal{T},\text{nc}}^{p}. \tag{36}$$

[1] Here, we use the observation that for a polynomial $q \in \mathbb{P}_p(\omega)$, $\omega \subset \Omega$ with positive measure, it holds either $q|_{\omega} = 0$ or $\operatorname{supp} q = \omega$. In our application we choose $q = \mathbf{e}_1 + \mathbf{e}_2$ and apply the argument simplex by simplex.

We say that the exact solution $\mathbf{e} \in \mathbf{L}^2(\Omega)$ is piecewise smooth over the partition $\mathcal{P} = (\Omega_j)_{j=1}^{J}$, if there exists some integer $s \geq 1$ such that

$$\mathbf{e}_{|\Omega_j} \in \mathbf{H}^s(\Omega_j) := (H^s(\Omega_j))^d \quad \text{for } j = 1, 2, \ldots, J.$$

We write $\mathbf{e} \in \mathbf{PH}^s(\Omega)$ and refer for further properties and generalizations to non-integer values of s, e.g., to [18, Sec. 4.1.9].

For the approximation results, the finite element meshes $\mathcal{T}$ are assumed to be compatible with the partition $\mathcal{P}$ in the following sense: for all $\tau \in \mathcal{T}$, there exists a single index j such that $\overset{\circ}{\tau} \cap \Omega_j \neq \emptyset$.

Theorem 15 *Let the electrostatic permeability ε satisfy assumptions (21), (22) and let $\rho \in L^2(\Omega)$. As an additional assumption on the regularity of the exact solution, we require that the exact solution of (4) satisfies $\mathbf{e} \in \mathbf{PH}^s(\Omega)$ for some integer $s \geq 1$. Assume that the non-conforming finite element space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ and the extended lifting operator $\Lambda_{\mathcal{T}}$ are defined on a compatible mesh $\mathcal{T}$, as in Definitions 11 and 13. Then, the non-conforming Galerkin discretization (36) has a unique solution which satisfies*

$$\|\mathbf{e} - \mathbf{e}_{\mathcal{T}}\|_{\mathbf{L}^2(\Omega)} \leq Ch^r \|\mathbf{e}\|_{\mathbf{PH}^r(\Omega)},$$

with $r := \min\{p+1, s\}$. The constant C only depends on $\varepsilon_{\min}$, $\varepsilon_{\max}$, $\|\varepsilon\|_{PW^{r,\infty}(\Omega)}$, p, and the shape regularity of the mesh.

Proof. The second Strang lemma applied to the non-conforming Galerkin discretization (36) implies the existence of a unique solution which satisfies the error estimate

$$\|\mathbf{e} - \mathbf{e}_{\mathcal{T}}\|_{\mathbf{L}^2(\Omega)} \leq \left(1 + \frac{\varepsilon_{\max}}{\varepsilon_{\min}}\right) \inf_{\tilde{\mathbf{e}} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}} \|\mathbf{e} - \tilde{\mathbf{e}}\|_{\mathbf{L}^2(\Omega)} + \frac{1}{\varepsilon_{\min}} \sup_{\tilde{\mathbf{e}} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \setminus \{0\}} \frac{|\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}})|}{\|\tilde{\mathbf{e}}\|_{\mathbf{L}^2(\Omega)}},$$

where

$$\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}}) := \int_\Omega \varepsilon \mathbf{e} \cdot \tilde{\mathbf{e}} - \int_\Omega \rho \Lambda_{\mathcal{T}} \tilde{\mathbf{e}}.$$

The approximation properties of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ in the infimum are inherited from the approximation properties of $\mathbf{E}^p_{\mathcal{T}}$ because of the inclusion $\mathbf{E}^p_{\mathcal{T}} \subset \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$; cf. (24). For the second term we obtain

$$\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}}) = \int_\Omega \varepsilon \mathbf{e} \cdot \nabla_{\mathcal{T}} \Lambda_{\mathcal{T}} \tilde{\mathbf{e}} - \int_\Omega \rho \Lambda_{\mathcal{T}} \tilde{\mathbf{e}}. \tag{37}$$

Note that $\rho \in L^2(\Omega)$ implies that $\operatorname{div}(\varepsilon \mathbf{e}) \in L^2(\Omega)$ and, in turn, that the jump $[\varepsilon \mathbf{e} \cdot \mathbf{n}_F]_F$ equals zero and the restriction $(\varepsilon \mathbf{e} \cdot \mathbf{n}_F)|_F$ is well defined for all $F \in \mathcal{F}$. We may apply simplexwise integration by parts to (37) to obtain

$$\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}}) = -\sum_{F \in \mathcal{F}} \int_F s_F \varepsilon (\mathbf{e} \cdot \mathbf{n}_F) [\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_F + \sum_{F \in \mathcal{F}_{\partial\Omega}} \int_F \varepsilon (\mathbf{e} \cdot \mathbf{n}_F) \Lambda_{\mathcal{T}} \tilde{\mathbf{e}}.$$

Above, $s_F = \pm 1$ depending on the orientation of the facet F.

Let $q_F \in \mathbb{P}^p_{d-1}(F)$ denote the best approximation of $\varepsilon \mathbf{e} \cdot \mathbf{n}_F|_F$ with respect to the $L^2(F)$ norm. Then, the combination of (31) with standard approximation properties and a trace inequality (since $r \geq 1$) leads to

$$\begin{aligned}
|\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}})| &= \left| -\sum_{F\in\mathcal{F}} \int_F s_F (\varepsilon \mathbf{e}\cdot \mathbf{n}_F - q_F) [\Lambda_{\mathcal{T}} \tilde{\mathbf{e}}]_F + \sum_{F\in\mathcal{F}_{\partial\Omega}} \int_F (\varepsilon \mathbf{e}\cdot \mathbf{n}_F - q_F) \Lambda_{\mathcal{T}} \tilde{\mathbf{e}} \right| \\
&\leq \sum_{F\in\mathcal{F}} \|\varepsilon \mathbf{e}\cdot \mathbf{n}_F - q_F\|_{L^2(F)} \|[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_F\|_{L^2(F)} \\
&\qquad\qquad\qquad + \sum_{F\in\mathcal{F}_{\partial\Omega}} \|\varepsilon \mathbf{e}\cdot \mathbf{n}_F - q_F\|_{L^2(F)} \|\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}\|_{L^2(F)} \\
&\leq C \left(\sum_{F\in\mathcal{F}} h_F^{r-1/2} \|\mathbf{e}\|_{H^r(\tau_F)} \|[\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}]_F\|_{L^2(F)} \right. \\
&\qquad\qquad\qquad \left. + \sum_{F\in\mathcal{F}_{\partial\Omega}} h_F^{r-1/2} \|\mathbf{e}\|_{H^r(\tau_F)} \|\Lambda_{\mathcal{T}}\tilde{\mathbf{e}}\|_{L^2(F)} \right),
\end{aligned}$$

where C depends only on p, s, and $\|\varepsilon\|_{W^r(\tau_F)}$, and the shape regularity of the mesh, and τ_F is one simplex among those of ω_F. The estimates (34),(35) along with the shape regularity of the mesh lead to the consistency estimate

$$\begin{aligned}
|\mathcal{L}_{\mathbf{e}}(\tilde{\mathbf{e}})| &\leq C \left(\sum_{F\in\mathcal{F}} h_F^r \|\mathbf{e}\|_{H^r(\tau_F)} \|\tilde{\mathbf{e}}\|_{L^2(\omega_F)} + \sum_{F\in\mathcal{F}_{\partial\Omega}} h_F^r \|\mathbf{e}\|_{H^r(\tau_F)} \|\tilde{\mathbf{e}}\|_{L^2(\omega_F)} \right) \\
&\leq \tilde{C} h^r \|\mathbf{e}\|_{PH^r(\Omega)} \|\tilde{\mathbf{e}}\|_{L^2(\Omega)},
\end{aligned}$$

which completes the proof. ∎

Remark 16 *If one chooses in (31) a degree $p' < p$ for the test-polynomials q, then the order of convergence behaves like $h^{r'} \|\mathbf{e}\|_{H^{r'}(\Omega)}$, with $r' := \min\{p'+1, s\}$, because the best approximation q_F now belongs to $\mathbb{P}^{p'}_{d-1}(F)$. Also, the above proof can be easily generalized to the case where $\mathbf{e} \in \mathbf{PH}^s(\Omega)$ for some $s > 1/2$.*

4.2 A local basis for non-conforming intrinsic finite elements in two dimensions

Like in Proposition 7, we construct the space $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ by defining basis functions whose supports are given by a single triangle $\tau \in \mathcal{T}$, facet-oriented basis functions whose supports are given by ω_F, $F \in \mathcal{F}$, and vertex-oriented basis functions whose supports are given by ω_V, $V \in \mathcal{V}$. The corresponding spaces are denoted by $\mathbf{B}^p_{\tau,\mathrm{nc}}$, $\mathbf{B}^p_{F,\mathrm{nc}}$, $\mathbf{B}^p_{V,\mathrm{nc}}$. The triangle-supported subspaces are given by

$$\mathbf{B}^p_{\tau,\mathrm{nc}} := \left\{ \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \mid \operatorname{supp} \mathbf{e} \subset \tau \right\} \qquad \forall \tau \in \mathcal{T}. \tag{38}$$

The definitions of $\mathcal{T}_F$, ω_F, $\mathcal{F}_V$, $\mathcal{T}_V$, ω_V are given in (16). The facet- and vertex-oriented subspaces will satisfy the following direct sum decompositions

$$\mathbf{B}^p_{F,\mathrm{nc}} \oplus \bigoplus_{\tau\in\mathcal{T}_F} \mathbf{B}^p_{\tau,\mathrm{nc}} = \left\{ \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \mid \operatorname{supp} \mathbf{e} \subset \omega_F \right\} \quad \forall F \in \mathcal{F}, \quad (39)$$

$$\mathbf{B}^p_{V,\mathrm{nc}} \oplus \bigoplus_{F\in\mathcal{F}_V} \mathbf{B}^p_{F,\mathrm{nc}} \oplus \bigoplus_{\tau\in\mathcal{T}_V} \mathbf{B}^p_{\tau,\mathrm{nc}} = \left\{ \mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \mid \operatorname{supp} \mathbf{e} \subset \omega_V \right\} \quad \forall V \in \mathcal{V}. \quad (40)$$

In Theorem 22, we will prove that $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ can be decomposed into a direct sum of these local subspaces.

4.2.1 Triangle-supported basis functions

In this section, let $\tau \in \mathcal{T}$ denote any fixed triangle in the mesh. The Lagrange basis of $\mathbb{P}^p_2(\tau)$ with respect to $\mathcal{N}^p \cap \tau$ is denoted by $b^\tau_{p,N}$, $N \in \mathcal{N}^p \cap \tau$, and is characterized by

$$b^\tau_{N,p} \in \mathbb{P}^p_2(\tau) \quad \text{and} \quad \forall N' \in \mathcal{N}^p \cap \tau \quad b^\tau_{N,p}(N') = \begin{cases} 1 & \text{if } N = N', \\ 0 & \text{if } N \neq N'. \end{cases} \quad (41)$$

We denote the (discontinuous in general) extension by zero of $b^\tau_{p,N}$ to $\Omega\backslash\tau$ again by $b^\tau_{p,N}$. From Lemma 4 and Conditions (24), (31), we deduce that

$$\mathbf{B}^p_{\tau,\mathrm{nc}} = \Big\{ \left.\mathbf{e}\right|_\tau \in \nabla\mathbb{P}^{p+1}_2(\tau) \mid \operatorname{supp} \mathbf{e} \subset \tau \quad \text{and} \\ \forall F \subset \partial\tau,\ \forall q \in \mathbb{P}^p_1(F) : \int_F \Lambda_{\mathcal{T}} \mathbf{e}\, q = 0 \Big\}. \quad (42)$$

According to (42), it is clear that $\mathbf{B}^p_\tau \subset \mathbf{B}^p_{\tau,\mathrm{nc}}$. In the next step, we use the weak compatibility conditions in (42) for the explicit characterization of $\mathbf{B}^p_{\tau,\mathrm{nc}}$.

For the construction of the non-conforming triangle-supported functions we have to introduce scaled versions of Legendre polynomials. For $F \in \mathcal{F} \cup \mathcal{F}_{\partial\Omega}$, let χ_F be the affine pullback to $[-1,1]$. Let $L_q : [-1,1] \to \mathbb{R}$ denote the Legendre polynomials of degree q with the normalization convention that $L_q(1) = 1$. This in turn implies that $L_q(-1) = (-1)^q$. We lift them to the facet F via $L^F_q := L_q \circ \chi_F^{-1}$. It is well known that L^F_{q+1} satisfies the orthogonality condition

$$(L^F_{q+1}, w)_{L^2(F)} = 0 \qquad \forall w \in \mathbb{P}^q_1(F). \quad (43)$$

Lemma 17 *For $\tau \in \mathcal{T}$, the non-conforming finite element space $\mathbf{B}^p_{\tau,\mathrm{nc}}$ is given by*

$$\mathbf{B}^p_{\tau,\mathrm{nc}} = \begin{cases} \mathbf{B}^p_\tau & \text{if } p \text{ is even}, \\ \mathbf{B}^p_\tau + \operatorname{span}\left\{\nabla_{\mathcal{T}} U^\tau_{p+1}\right\} & \text{if } p \text{ is odd}, \end{cases} \quad (44)$$

where U^τ_{p+1} is defined as follows. For any $N \in \mathcal{N}^{p+1} \cap \partial\tau$, let $F_N \subset \partial\tau$ denote a fixed, but arbitrary, facet such that $N \in F_N$. Then U^τ_{p+1} is given by

$$U^\tau_{p+1} := \sum_{N\in\mathcal{N}^{p+1}\cap\partial\tau} L^{F_N}_{p+1}(N)\, b^\tau_{p+1,N} \quad (45)$$

and illustrated for $p = 3, 5$ in Figure 1.

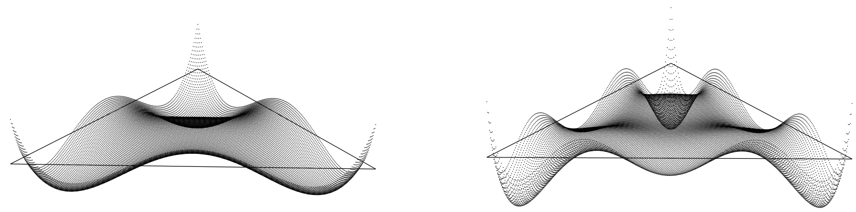

Figure 1: Representation of U_{p+1}^{τ} for $p = 3$ (left) and $p = 5$ (right).

Proof. Pick some $\mathbf{e} \in \mathbf{B}_{\tau,\mathrm{nc}}^{p}$, let $u := \Lambda_{\mathcal{T}}\mathbf{e}$ (according to Proposition 14, $\operatorname{supp} u \subset \tau$) and denote the restrictions to τ by $\mathbf{e}_\tau$ and u_τ. The weak compatibility condition in (42) therefore implies that, for all $F \subset \partial\tau$,

$$u_\tau|_F = c_F L_{p+1}^F \quad \text{for some } c_F \in \mathbb{R}. \tag{46}$$

The relation $u_\tau \in \mathbb{P}_2^{p+1}(\tau)$ implies that $u_\tau|_{\partial\tau}$ is continuous so that u_τ is continuous at every vertex of τ. We distinguish two cases.

Let p be even. In this case we have $L_{p+1}(1) = -L_{p+1}(-1)$ so that the continuity at the vertices of τ implies $c_F = 0$. Thus $u_\tau|_{\partial\tau} = 0$ and we have proved (44) for even p.

Let p be odd. Now we have $L_{p+1}(1) = L_{p+1}(-1)$ so that $c_F = c_\tau$ for all $F \subset \partial\tau$ and some fixed c_τ, and we conclude that $u_\tau = c_\tau U_{p+1}^{\tau}$, with U_{p+1}^{τ} given in (45). Conversely, we note that the gradient of U_{p+1}^{τ} satisfies the weak compatibility condition (31). This leads to the assertion for odd p. ∎

Remark 18 *A basis of* $\mathbf{B}_{\tau,\mathrm{nc}}^{p}$ *for even* p *is given by* $\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}\right\}$, *while a basis for odd* p *is given by* $\left\{\nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \overset{\circ}{\tau}\right\} \cup \left\{\nabla_{\mathcal{T}} U_{p+1}^{\tau}\right\}$.

4.2.2 Facet-oriented basis functions

Lemma 19 *For* $F \in \mathcal{F}$, *the non-conforming finite element space* $\mathbf{B}_{F,\mathrm{nc}}^{p}$ *that satisfies (39) is given by*

$$\mathbf{B}_{F,\mathrm{nc}}^{p} = \begin{cases} \mathbf{B}_F^p + \operatorname{span}\left\{\nabla_{\mathcal{T}} U_{p+1}^F\right\} & \text{if } p \text{ is even,} \\ \mathbf{B}_F^p & \text{if } p \text{ is odd,} \end{cases} \tag{47}$$

where U_{p+1}^F *is defined as follows. For* $N \in \mathcal{N}^{p+1} \cap \partial\omega_F$, *let*

$$b_{p+1,N}^F := \begin{cases} b_{p+1,N}^{\mathcal{T}}|_{\omega_F} & \text{on } \omega_F, \\ 0 & \text{on } \Omega\backslash\omega_F, \end{cases} \tag{48}$$

where $b^{\mathcal{T}}_{p+1,N}$ are as in (15). Then, U^F_{p+1} is given by

$$U^F_{p+1} := \sum_{N\in\mathcal{N}^{p+1}\cap\partial\omega_F} L^{F_N}_{p+1}(N)\, b^F_{p+1,N} \tag{49}$$

with the lifted Legendre polynomials satisfying (43) and where, for $N\in\mathcal{N}^{p+1}\cap\partial\omega_F$, we assign some facet $F_N\subset\partial\omega_F$ such that $N\in F_N$.

Proof. For $F\in\mathcal{F}$, given $\mathbf{e}\in\mathbf{B}^p_F$, it follows from Definition 6 that $\operatorname{supp}\mathbf{e}\subset\omega_F$, without being supported on only one triangle (otherwise, $\mathbf{e}\in\mathbf{B}^p_\tau$ for some $\tau\in\mathcal{T}_F$). Then it follows from conditions (38) and (39) that $\mathbf{e}\in\mathbf{B}^p_{F,\mathrm{nc}}$. Hence $\mathbf{B}^p_F\subset\mathbf{B}^p_{F,\mathrm{nc}}$. Since any $\mathbf{e}\in\mathbf{B}^p_{F,\mathrm{nc}}$ can be expressed locally on $\tau\in\mathcal{T}_F$ by $\mathbf{e}|_\tau=\nabla v_\tau$ for some $v_\tau\in\mathbb{P}^{p+1}_2(\tau)$ (cf. Lemma 4) we have

$$\mathbf{B}^p_{F,\mathrm{nc}}\subset\bigoplus_{\tau\in\mathcal{T}_F}\operatorname{span}\left\{\nabla_{\mathcal{T}} b^\tau_{N,p+1}\mid N\in\mathcal{N}^{p+1}\cap\tau\right\},$$

where we recall (cf. (41)) that $b^\tau_{N,p+1}$ are the Lagrange basis functions on τ and extended by zero to $\Omega\backslash\tau$. Since the functions $b^\tau_{N,p+1}$ for the inner nodes $N\in\mathcal{N}^{p+1}\cap\mathring{\tau}$ belong to the space $\mathbf{B}^p_\tau\subset\mathbf{B}^p_{\tau,\mathrm{nc}}$, we conclude from (39) that

$$\mathbf{B}^p_{F,\mathrm{nc}}\subset\bigoplus_{\tau\in\mathcal{T}_F}\operatorname{span}\left\{\nabla_{\mathcal{T}} b^\tau_{N,p+1}\mid N\in\mathcal{N}^{p+1}\cap\partial\tau\right\}.$$

For $\mathbf{e}\in\mathbf{B}^p_{F,\mathrm{nc}}$, let $u:=\Lambda_{\mathcal{T}}\mathbf{e}$ ($\operatorname{supp}u\subset\omega_F$, cf. Proposition 14) and $u_\tau:=u|_\tau$, $\tau\in\mathcal{T}_F$. Arguing as in the case of triangle-supported basis functions, we infer from the compatibility conditions (31)

$$[u]_F=c_F L^F_{p+1}\quad\text{and}\quad\forall F'\subset\partial\omega_F\quad u|_{F'}=c_{F'}L^{F'}_{p+1}. \tag{50}$$

Again, the relation $u_\tau\in\mathbb{P}^{p+1}_2(\tau)$ implies the continuity of u_τ at the vertices of τ. Using this property, we now split the proof into two parts. In the following we identify a space $\mathbf{R}^p_{F,\mathrm{nc}}$ which satisfies

$$\mathbf{B}^p_{F,\mathrm{nc}}=\mathbf{B}^p_F\oplus\mathbf{R}^p_{F,\mathrm{nc}}. \tag{51}$$

Let p be even. For $\tau\in\mathcal{T}_F$, the continuity of u_τ along $\partial\tau$ and the endpoint properties of $L^{F'}_{p+1}$ imply that $u_\tau(A^F)=u_\tau(B^F)$, where A^F,B^F denote the endpoints of F (cf. Figure 2). Hence, $[u]_F(A^F)=[u]_F(B^F)$. Since $L^F_{p+1}(A^F)=-L^F_{p+1}(B^F)$ we conclude that the first condition in (50) holds with $c_F=0$: in other words, u is continuous across F.

The results obtained so far imply that

$$\mathbf{R}^p_{F,\mathrm{nc}}\subset\operatorname{span}\left\{\nabla_{\mathcal{T}} b^F_{p+1,N}\mid N\in\mathcal{N}^{p+1}\cap\partial\omega_F\right\}.$$

Pick $\mathbf{e}\in\mathbf{R}^p_{F,\mathrm{nc}}$ and set $u:=\Lambda_{\mathcal{T}}\mathbf{e}$. The continuity property $[u]_F=0$ which we already derived implies $u=cU^F_{p+1}$ with U^F_{p+1} given in (49). On the other

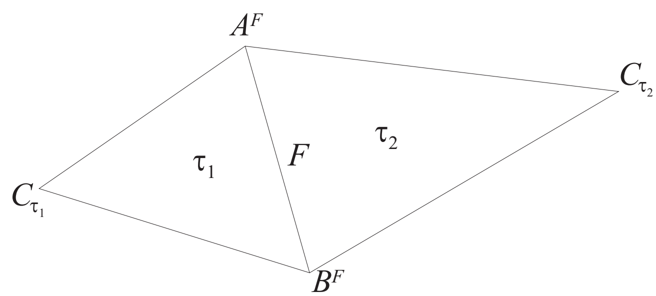

Figure 2: A face $F \in \mathcal{F}$ with endpoints A^F, B^F and two neighboring triangles τ_1, τ_2,

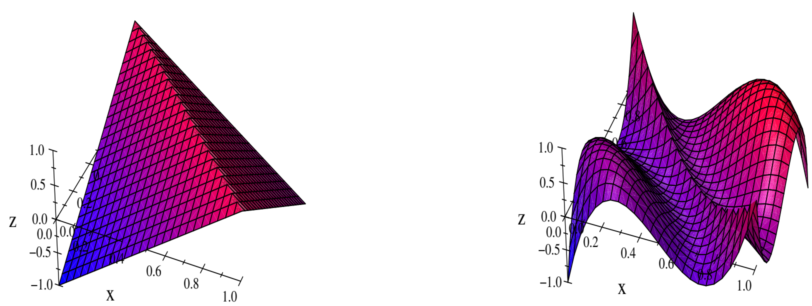

Figure 3: The non-conforming basis functions U_{p+1}^F have support on two adjacent triangles and are depicted for $p = 0$ (left) and $p = 2$ (right).

hand, $\nabla_{\mathcal{T}} U_{p+1}^F$ fulfills the weak compatibility conditions (31). Hence we may set $\mathbf{R}_{F,\mathrm{nc}}^p := \operatorname{span}\left\{\nabla_{\mathcal{T}} U_{p+1}^F\right\}$ and the assertion follows for even p. The functions U_{p+1}^F for $p = 0$ and $p = 2$ are depicted in Figure 3.

Let p be odd. Pick $\mathbf{e} \in \mathbf{R}_{F,\mathrm{nc}}^p$ and set $u := \Lambda_{\mathcal{T}} \mathbf{e}$. For $\tau \in \mathcal{T}_F$ and any facet $F' \subset \partial\omega_F \cap \partial\tau$, the restriction of u_τ to F' must be a multiple of a Legendre polynomial. The continuity of u_τ along $\partial\tau$ implies in particular the continuity at C_τ (cf. Figure 2). Hence, $u_\tau|_{\partial\omega_F \cap \partial\tau} = c_\tau U_{p+1}^\tau|_{\partial\omega_F \cap \partial\tau}$ for some c_τ and U_{p+1}^τ as defined in (45), and

$$\tilde{u} = u - \sum_{\tau \in \mathcal{T}_F} c_\tau U_{p+1}^\tau$$

vanishes along $\partial\omega_F$ with $\operatorname{supp} \tilde{u} \subset \omega_F$. So the jump of $\tilde{u}$ across F vanishes in A^F and B^F, and the expression of the first condition in (50) is written as $[\tilde{u}]_F = 0$. Hence $\tilde{u}$ is continuous in ω_F and vanishes on $\partial\omega_F$. From this we conclude that

$\tilde{u} \in \mathbf{B}_F^p$ (see Definition 6). The characterization of $\mathbf{R}_{F,\mathrm{nc}}^p$ as a direct sum in (51) shows that $u = 0$ and thus $\mathbf{R}_{F,\mathrm{nc}}^p = \{0\}$. ■

Remark 20 *A basis of* $\mathbf{B}_{F,\mathrm{nc}}^p$ *for odd* p *is given by* $\left\{ \nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \mathring{F} \right\}$ *while for even* p *we may choose* $\left\{ \nabla_{\mathcal{T}} b_{p+1,N}^{\mathcal{T}} : N \in \mathcal{N}^{p+1} \cap \mathring{F} \right\} \cup \left\{ \nabla_{\mathcal{T}} U_{p+1}^F \right\}$.

4.2.3 Vertex-oriented basis functions

In this section we now identify the vertex-oriented subspace $\mathbf{B}_{V,\mathrm{nc}}^p$.

Lemma 21 *Let* $V \in \mathcal{V}$. *It holds*

$$\mathbf{B}_{V,\mathrm{nc}}^p = \begin{cases} \{0\} & \text{if } p \text{ is even,} \\ \mathbf{B}_V^p & \text{if } p \text{ is odd.} \end{cases} \tag{52}$$

Proof. In a first step, we will prove that any subspace $\mathbf{R}_{p+1,V}^{\mathcal{T}}$ which satisfies the direct sum decomposition

$$\mathbf{R}_{p+1,V}^{\mathcal{T}} \oplus \bigoplus_{F \in \mathcal{F}_V} \mathbf{B}_{F,\mathrm{nc}}^p \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^p = \left\{ \mathbf{e}' \in \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \middle| \operatorname{supp} \mathbf{e}' \subset \omega_V \right\}, \tag{53}$$

also satisfies

$$\mathbf{R}_{p+1,V}^{\mathcal{T}} \subset \mathbf{B}_V^p. \tag{54}$$

We recall from Definition 6 that $\mathbf{B}_V^p = \operatorname{span}\left\{ \nabla b_{p+1,V}^{\mathcal{T}} \right\}$.

In the second step, we will show that, for even p the inclusion

$$\operatorname{span}\left\{ \nabla b_{p+1,V}^{\mathcal{T}} \right\} \subset \bigoplus_{F \in \mathcal{F}_V} \mathbf{B}_{F,\mathrm{nc}}^p \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^p \tag{55}$$

holds so that the first case in (52) follows.

Instead, for odd p, we will prove that, for all $V \in \mathcal{V}$,

$$\nabla b_{p+1,V}^{\mathcal{T}} \notin \bigoplus_{F \in \mathcal{F}_V} \mathbf{B}_{F,\mathrm{nc}}^p \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^p. \tag{56}$$

From (40) and (54), we conclude that the second case of (52) follows.

1st Step: Choose any

$$\mathbf{e} \in \left\{ \mathbf{e}' \in \mathbf{E}_{\mathcal{T},\mathrm{nc}}^p \middle| \operatorname{supp} \mathbf{e}' \subset \omega_V \right\} \tag{57}$$

and set $u := \Lambda_{\mathcal{T}} \mathbf{e}$. According to Proposition 14, $\operatorname{supp} u \subset \omega_V$.

Let p **be odd.** For $\tau \in \mathcal{T}_V$, the facet F^τ is defined by the condition $F^\tau \subset \partial\tau \cap \partial\omega_V$ (cf. Figure 4). Since $L_{p+1}^{F^\tau}$ has even degree, the values at the endpoints A^τ, B^τ of F^τ are both equal to one.

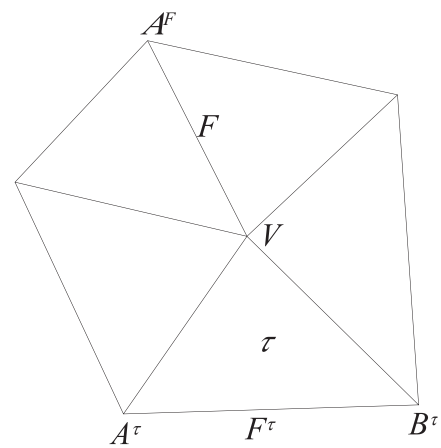

Figure 4: A vertex $V \in \mathcal{V}$, a neighboring triangle $\tau \in \mathcal{T}_V$, and a neighboring facet $F \in \mathcal{T}_V$.

We set $u_\tau := u|_{\overset{\circ}{\tau}}$ and define (cf. (45))

$$\tilde{u} := u - \sum_{\tau \in \mathcal{T}_V} u_\tau \left(A^\tau\right) U_{p+1}^\tau \quad \text{with} \quad u_\tau \left(A^\tau\right) := \lim_{\substack{x \to A^\tau \\ x \in \overset{\circ}{\tau}}} u_\tau \left(x\right),$$

where the sum over the triangles is well defined in the interior of the triangles as well as the one-sided traces from the interior of a triangle towards its boundary.

Hence, $\tilde{u} = 0$ on $\partial\omega_V$ with $\operatorname{supp} \tilde{u} \subset \omega_V$. Any facet $F \in \mathcal{F}_V$ has V as one endpoint; denote the other one by A^F. According to the weak compatibility conditions, we know that $[\tilde{u}]_F$ is proportional to L_{p+1}^F on any facet $F \in \mathcal{F}_V$. Then, we use the condition $[\tilde{u}]_F = c_F L_{p+1}^F$ at the point A^F to deduce $c_F = 0$ from $\tilde{u}|_{\partial\omega_V} = 0$. Hence $\tilde{u}$ is continuous and vanishes on $\partial\omega_V$. Consequently, $\tilde{u}$ is a conforming function, i.e.,

$$\begin{aligned} \nabla \left(u - \sum_{\tau \in \mathcal{T}_V} u_\tau \left(A^\tau\right) U_{p+1}^\tau \right) &\in \mathbf{B}_V^p \oplus \bigoplus_{F \in \mathcal{F}_V} \mathbf{B}_F^p \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}_\tau^p \\ &\subset \mathbf{B}_V^p \oplus \bigoplus_{F \in \mathcal{F}_V} \mathbf{B}_{F,\mathrm{nc}}^p \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}_{\tau,\mathrm{nc}}^p. \end{aligned}$$

Hence, (53) implies $\mathbf{R}_{p+1,V}^{\mathcal{T}} \subset \mathbf{B}_V^p$.

Let p be even. We number the facets in $\mathcal{F}_V$ counter-clockwise as $\mathcal{F}_V = \{F_1, \ldots, F_q\}$ (see Figure 5) for some q and, to simplify the notation, we set $F_0 := F_q$ and $F_{q+1} := F_1$. The triangle which has F_{i-1} and F_i as facets and V as a vertex is denoted by τ_i. Each facet F_i has V as an endpoint; denote by $\mathbf{A}_i$ the other one. We further set $F_i^{\mathrm{out}} := \partial\tau_i \cap \partial\omega_V$. We define recursively $u_0 := u$

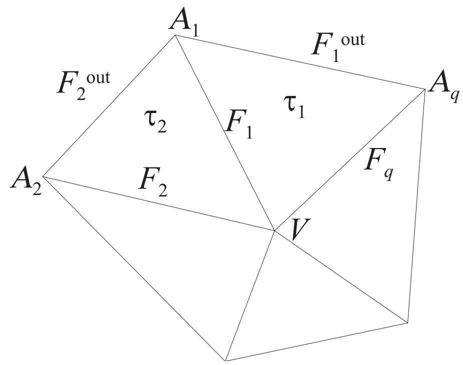

Figure 5: A vertex $V \in \mathcal{V}$ and its outgoing facets numbered counterclockwise. The triangles $\tau_i \in \mathcal{T}_V$ contain F_{i-1}, F_i, F_i^{out} as facets and V as a vertex.

and, for $k = 1, 2, \ldots, q$, (cf. (49))

$$u_k = u_{k-1} - \frac{(u_{k-1})_{\tau_k}(\mathbf{A}_k)}{U_{p+1}^{F_k}(\mathbf{A}_k)} U_{p+1}^{F_k} \quad \text{with} \quad (u_{k-1})_{\tau_k}(\mathbf{A}_k) := \lim_{\substack{x \to \mathbf{A}_k \\ x \in \overset{\circ}{\tau}_k}} u_{k-1}(x).$$

Note that $u_q = 0$ on $\partial\omega_V \backslash F_1^{\text{out}}$. All functions u_k are supported in ω_V. Arguing as for the case of odd p we deduce that u_q is continuous on $\omega_V \backslash F_1^{\text{out}}$. Next, we define the non-conforming part of u_q by $u_q^+ := u_q - \sum_{N \in \mathcal{N}^{p+1} \backslash \{F_1^{\text{out}}\}} u_q(N)\, b_{p+1,N}^{\mathcal{T}}$. It follows that $\operatorname{supp} u_q^+ \subset \tau_1$ and hence $u_q^+ \in \mathbf{B}_{\tau_1,\text{nc}}^p$. For even p, we have proved $\mathbf{B}_{\tau_1,\text{nc}}^p = \mathbf{B}_{\tau_1}^p$, so that u_q^+ must be continuous on Ω. As $\sum_{N \in \mathcal{N}^{p+1} \backslash \{F_1^{\text{out}}\}} u_q(N)\, b_{p+1,N}^{\mathcal{T}}$ is also continuous on Ω, so is u_q. In particular, this yields that u_q is continuous on ω_V and the assertion follows as in the case of odd p. We conclude again that $\mathbf{R}_{p+1,V}^{\mathcal{T}} \subset \mathbf{B}_V^p$.

2nd Step: To prove (55) and (56) we again distinguish between even and odd values of p.

Let p be even. We employ the same notation as in the 1st step for the case p even. Then, by using U_{p+1}^F as in (49) and recalling that U_{p+1}^F is continuous across F, we define a function

$$w_1 := b_{p+1,V}^{\mathcal{T}} - \frac{1}{q}\sum_{i=1}^{q} W_i \quad \text{with} \quad W_i := \left(\lim_{\substack{x \to V \\ x \in F_i}} U_{p+1}^{F_i}(x)\right) U_{p+1}^{F_i}. \tag{58}$$

By construction, $\operatorname{supp} w_1 \subset \omega_V$. Let us consider a fixed facet F_i. Note that the functions $U_{p+1}^{F_j}$ are continuous across F_i for $j \notin \{i-1, i+1\}$. However, the one-sided limits for W_{i-1} and W_{i+1} at F_i coincide so that w_1 is continuous in ω_V and vanishes at V and at all inner nodes $\mathcal{N}^{p+1} \cap \overset{\circ}{\tau}$, $\tau \in \mathcal{T}_V$. On the other hand, the function w_1 is determined on some outer facet F_i^{out} by two consecutive terms in the sum in (58), i.e.,

$$w_1|_{F_i^{\text{out}}} = (W_{i-1} + W_i)|_{F_i^{\text{out}}}.$$

Note that $W_{i-1}(V) = W_i(V) = 1$ considered as limit values along the facets F_{i-1}, F_i. The sign properties of a facet-oriented basis function for even p implies that W_{i-1} has value 1 at $\mathbf{A}_{i-1}$ and value -1 at $\mathbf{A}_i$. Vice versa, W_i has value -1 at $\mathbf{A}_{i-1}$ and value 1 at $\mathbf{A}_i$. Hence, $w_1|_{F_i^{\text{out}}}$ is a Legendre polynomial with endpoints values 0 which implies $w_1|_{F_i^{\text{out}}} = 0$ and, in turn, $w_1 = 0$ on $\partial\omega_V$. Up to now, we have thus proved that w_1 is continuous in Ω, with support contained in ω_V and value 0 at V and at all nodal points $\mathring{\tau}_i \cap \mathcal{N}^{p+1}$.

Next we define

$$w_2 = w_1 - \sum_{i=1}^{q} \sum_{N \in \mathcal{N}^{p+1} \cap \mathring{F}_i} w_1(N)\, b_{p+1,N}^{\mathcal{T}} \tag{59}$$

and observe that w_2 is a conforming function which vanishes at all nodal points in $\mathcal{N}^{p+1}$. This implies that $w_2 = 0$ in Ω and we have established (55), or, more precisely, that

$$\nabla b_{p+1,V}^{\mathcal{T}} \in \bigoplus_{F \in \mathcal{F}_V} \mathbf{B}_{F,\text{nc}}^{p}.$$

Let p be odd. We will prove (56) by contradiction. So, assume that

$$\nabla b_{p+1,V}^{\mathcal{T}} \in \bigoplus_{F \in \mathcal{F}_V} \mathbf{B}_{F,\text{nc}}^{p} \oplus \bigoplus_{\tau \in \mathcal{T}_V} \mathbf{B}_{\tau,\text{nc}}^{p}.$$

We then infer from Remark 18 and Remark 20 that

$$b_{p+1,V}^{\mathcal{T}} = \sum_{N \in \mathcal{N}^{p+1} \backslash \mathcal{V}} \alpha_N b_{p+1,N}^{\mathcal{T}} + \sum_{\tau \in \mathcal{T}} \alpha_\tau U_{p+1}^{\tau} \tag{60}$$

for some coefficients α_N and α_τ. Let $v_{\text{c}} := \sum_{N \in \mathcal{N}^{p+1} \backslash \mathcal{V}} \alpha_N b_{p+1,N}^{\mathcal{T}}$ and $v_{\text{nc}} := \sum_{\tau \in \mathcal{T}} \alpha_\tau U_{p+1}^{\tau}$. Since $b_{p+1,N}^{\mathcal{T}}$ and v_{c} are continuous in Ω, the function v_{nc} must also be continuous. By contradiction it is easy to prove that

$$C^0(\Omega) \cap \bigoplus_{\tau \in \mathcal{T}} \operatorname{span}\left\{U_{p+1}^{\tau}\right\} = \operatorname{span}\{U_{p+1}\} \text{ with } U_{p+1} := \sum_{\tau \in \mathcal{T}} U_{p+1}^{\tau},$$

so that $v_{\text{nc}} \in \operatorname{span}\{U_{p+1}\}$. Since $v_{\text{c}}(V) = 0$ and $b_{p+1,V}^{\mathcal{T}}(V) = 1$, we obtain from (60) that $v_{\text{nc}}(V) = 1$. The restriction of U_{p+1} to any facet $F \in \mathcal{F} \cup \mathcal{F}_{\partial\Omega}$ is a Legendre polynomial of even degree, which implies that $v_{\text{nc}}(V') = 1$, for every $V' \in \mathcal{V} \cup \mathcal{V}_{\partial\Omega}$. But the functions $b_{p+1,V}^{\mathcal{T}}$ and v_{c} vanish on $\partial\Omega$. This contradicts $v_{\text{nc}}(V') = 1$ for the boundary points $V' \in \mathcal{V}_{\partial\Omega}$. ∎

4.2.4 Properties of the non-conforming intrinsic basis functions

Theorem 22 *A basis of* $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ *is given by*

$$\left\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1} \backslash \mathcal{V}\right\} \cup \bigcup_{F\in\mathcal{F}} \left\{\nabla_{\mathcal{T}} U^F_{p+1}\right\} \quad \text{if } p \text{ is even,} \tag{61}$$

or by

$$\left\{\nabla_{\mathcal{T}} b^{\mathcal{T}}_{p+1,N} : N \in \mathcal{N}^{p+1}\right\} \cup \bigcup_{\tau\in\mathcal{T}} \left\{\nabla_{\mathcal{T}} U^\tau_{p+1}\right\} \quad \text{if } p \text{ is odd.} \tag{62}$$

Remark 23 *At first glance, it seems that* $\mathbf{B}^p_V \not\subset \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ *for even* p*. Actually, this subspace of* $\mathbf{E}^p_{\mathcal{T}}$ *has already been taken into account; see (55).*

Proof. By construction, the space $\widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$ of the functions found in (61) as in (62) is a subspace of $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$. So, it remains to prove that $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}} \subset \widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$.

Let p **be odd.** The arguments are very similar to those in the proof of Lemma 21 for odd p. Given $\mathbf{e} \in \mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$, let $u := \Lambda_{\mathcal{T}}\mathbf{e}$. Pick some $\tau \in \mathcal{T}$ having at least one facet on $\partial\Omega$. Condition (31) implies that, for all facets $F \subset \partial\tau \cap \partial\Omega$, the restriction $u|_F$ is a multiple of the lifted Legendre polynomial L^F_{p+1}. The continuity of $u|_\tau$ on τ implies that there exists a function $\tilde{u} := cU^\tau_{p+1}$ with $\nabla\tilde{u} \in \mathbf{B}^p_{\tau,\mathrm{nc}}$ for some c such that $u_1 := u - \tilde{u}$ satisfies $u_1|_{\partial\tau\cap\partial\Omega} = 0$. Since u_1 vanishes at the endpoints of all such facets $F \in \mathcal{F}_{\partial\Omega}$, the function u_1 is also continuous across the other facets $F \subset \partial\tau \cap \Omega$. Let

$$\begin{aligned}\tilde{u}_1 := \sum_{N\in\mathcal{N}^{p+1}\cap\mathring{\tau}} u_1(N)\, b^{\mathcal{T}}_{p+1,N} + \sum_{F\subset\partial\tau\cap\Omega} \sum_{N\in\mathcal{N}^{p+1}\cap\mathring{F}} u_1(N)\, b^{\mathcal{T}}_{p+1,N} \\ + \sum_{V\in\partial\tau\cap\Omega} u_1(V)\, b^{\mathcal{T}}_{p+1,V}\end{aligned}$$

and note that $\tilde{u}_1 \in \widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$, because Lemma 21 implies in particular that $b^{\mathcal{T}}_{p+1,V} \in \widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$. Note that $u_2 := u_1 - \tilde{u}_1$ vanishes on τ. Since Ω is connected, iterating this construction for the remaining triangles finally results in a function that vanishes on Ω, which yields a linear representation of u by functions in $\widetilde{\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}}$.

Let p **be even.** Again the arguments are very similar to those in the proof of Lemma 21 for even p. We omit the details here. ■

Remark 24 *Let* $\mathtt{V}, \mathtt{F}, \mathtt{T}$ *denote respectively the number of vertices, facets and triangles of the mesh. According to Euler's formula, one has* $\mathtt{V} - \mathtt{F} + \mathtt{T} = 1$ *because* Ω *has no holes (its boundary is connected). Also, if one splits* $\mathtt{V}$ *and* $\mathtt{F}$ *respectively into* $\mathtt{V} = \mathtt{V}_{int} + \mathtt{V}_{bdry}$ *and* $\mathtt{F} = \mathtt{F}_{int} + \mathtt{F}_{bdry}$*, with* $_{int}$ *denoting interior vertices and facets and* $_{bdry}$ *denoting boundary vertices and facets, one has* $\mathtt{V}_{bdry} = \mathtt{F}_{bdry}$*. Then the dimension of the vector space* $\mathbf{E}^p_{\mathcal{T},\mathrm{nc}}$ *is given by:*

for even p: $|\mathcal{N}^{p+1}| - \mathtt{V}_{int} + \mathtt{F}_{int} = |\mathcal{N}^{p+1}| - \mathtt{V} + \mathtt{F} = |\mathcal{N}^{p+1}| + \mathtt{T} - 1\,;$
for odd p: $|\mathcal{N}^{p+1}| + \mathtt{T}$.

As an illustration, let us consider non-conforming intrinsic basis functions of degree 0.

Proposition 25 *The lowest order non-conforming intrinsic finite elements are given by*

$$\mathbf{E}^{0}_{\mathcal{T},\mathrm{nc}} = \operatorname{span}\left\{\nabla_{\mathcal{T}} U_1^F : F \in \mathcal{F}\right\},$$

where the functions U_1^F are the standard non-conforming Crouzeix-Raviart basis functions (cf. [12]).

Proof. Choosing $p = 0$ and taking into account that $\mathcal{N}^1 = \mathcal{V}$ we conclude from (61) that a basis for $\mathbf{E}^{0}_{\mathcal{T},\mathrm{nc}}$ is given by $\bigcup_{F\in\mathcal{F}} \left\{\nabla_{\mathcal{T}} U_1^F\right\}$.

To show the connection with the Crouzeix-Raviart basis functions, we consider a facet $F \in \mathcal{F}$ with neighboring triangles τ_1 and τ_2. From (49), we deduce that U_1^F is affine on each of the triangles τ_1, τ_2 with value 1 at the endpoints of F and value -1 at the vertices of τ_1, τ_2 that are opposite to F. Hence, U_1^F coincides with the standard Crouzeix-Raviart basis functions; see again [12]. ∎

4.3 An example of a non-conforming intrinsic finite element in three dimensions

Although the general theory of non-conforming intrinsic finite elements in the form of Theorem 15 holds for $d = 2, 3$, the *construction* of a local basis requires further investigation which will be the topic of the forthcoming paper [10]. We emphasize that our theory allows to enrich a conforming finite element space by new, locally supported, non-conforming polynomials in a flexible way. In addition, for a given order of approximation, the number of non-conforming basis functions increases with the spatial dimension.

As an example we give here the definition of a non-conforming, simplex-supported basis function for $d = 3$: for $p \in \mathbb{N}_0$ and $0 \leq k \leq p$, define $b_{p,k} \in \mathbb{P}_2^p(\hat{\tau}_2)$ with $\hat{\tau}_2$ as in (13) by

$$b_{p,k}(\hat{x}_1, \hat{x}_2) := (\hat{x}_1 + \hat{x}_2)^k \, P_{p-k}^{(0,2k+1)}\left(2(\hat{x}_1 + \hat{x}_2) - 1\right) P_k^{(0,0)}\left(\frac{\hat{x}_1 - \hat{x}_2}{\hat{x}_1 + \hat{x}_2}\right) \quad \forall (\hat{x}_1, \hat{x}_2) \in \hat{\tau}_2,$$

where $P_p^{(\alpha,\beta)}$ are the Jacobi polynomials (see, e.g., [1, §22.3]) and let

$$f_{2\mathrm{D}} : \hat{\tau}_2 \to \mathbb{R} \qquad f_{2\mathrm{D}} := \sum_{k=0}^{3} \alpha_k b_{6,2k} \quad \text{with} \quad \alpha_0 = 3,\ \alpha_1 = 7,\ \alpha_2 = 0,\ \alpha_3 = 11.$$

The function $f_{2\,\mathrm{D}}$ has symmetry of order three, i.e., is invariant under affine bijections from $\hat{\tau}_2$ onto $\hat{\tau}_2$. As a consequence the function $f_{3\mathrm{D}} \in C^0(\partial\hat{\tau}_3)$, which is generated by lifting $f_{2\mathrm{D}}$ to the facets of $\partial\hat{\tau}_3$ via affine pullbacks to $\hat{\tau}_2$ (see Figure 6), is continuous. Then, $U_6^{\hat{\tau}_3}$ is generated by interpolating the function $f_{3\mathrm{D}}$ to the interior of $\hat{\tau}_3$ in an analogous fashion as explained for $d = 2$ in (45).

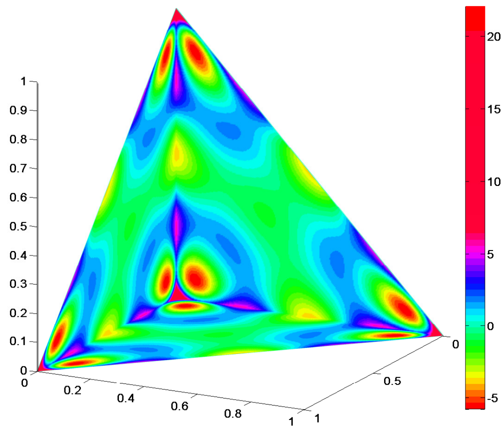

Figure 6: Surface plot of the non-conforming function $U_6^{\hat{\tau}_3}$. The support of this function is the unit simplex.

5 Conclusions

In this article we have developed a general method for constructing finite element spaces from intrinsic conforming and non-conforming conditions. As a model problem we have considered the Poisson equation, but this approach is by no means limited to this model problem. Using theoretical conditions in the spirit of the second Strang lemma, we have derived conforming and non-conforming finite element spaces of arbitrary order for the fluxes. For these spaces, we have also derived sets of local basis functions.

In two dimensions, it turns out that the lowest order non-conforming space is spanned by the trianglewise gradients of the standard non-conforming Crouzeix-Raviart basis functions. In general, all polynomial non-conforming spaces are spanned by the gradients of standard *hp*-finite element basis functions *enriched* by some facet-oriented non-conforming basis functions for even polynomial degree and by some triangle-supported non-conforming basis functions for odd polynomial degree. As a by-product, this methodology allowed us to recover the well-known non-conforming Crouzeix-Raviart element (cf. Proposition 25). By using a similar but more technical argument (cf. [20]), it can be shown that our intrinsic derivation of non-conforming finite elements also allows one to recover the second order non-conforming Fortin-Soulie element [13, 14], the third order Crouzeix-Falk element [11], and the family of Gauss-Legendre elements [4], [21]. In three dimensions, one may also use the same method: see Section 4.3 for an illustration. More systematic studies will be presented in the forthcoming paper [10].

In the past, the construction of a new finite element was an “art” and came, typically, before the development of its theory. Here, we have considered the construction of conforming and non-conforming finite elements and their analysis through a *unified approach*, and we have constructed all conforming and non-conforming, local and polynomial finite element element spaces which can be handled within the theory based on the second Strang lemma. In this respect the approach is similar in its spirit to the exterior calculus for finite elements in combination with their numerical stability analysis (see [3] and references therein). It is a topic of future research to investigate how our approach for non-conforming finite elements can be used for the development of an exterior calculus for non-conforming finite elements.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions.* Applied Mathematics Series 55. National Bureau of Standards, U.S. Department of Commerce, 1972.

[2] C. Amrouche and V. Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension. *Czechoslovak Mathematical Journal*, 44(1):109–140, 1994.

[3] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc. (N.S.)*, 47(2):281–354, 2010.

[4] Á. Baran and G. Stoyan. Gauss-Legendre elements: a stable, higher order non-conforming finite element family. *Computing*, 79(1):1–21, 2007.

[5] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*, volume 15. Springer-Verlag, 1991.

[6] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, 1978.

[7] P. G. Ciarlet and P. Ciarlet, Jr. Another approach to linearized elasticity and a new proof of Korn's inequality. *Math. Models Methods Appl. Sci.*, 15(2):259–271, 2005.

[8] P. G. Ciarlet and P. Ciarlet, Jr. A new approach for approximating linear elasticity problems. *C. R. Math. Acad. Sci. Paris Ser. I*, 346(5-6):351–356, 2008.

[9] P. G. Ciarlet and P. Ciarlet, Jr. Direct computation of stresses in planar linearized elasticity. *Math. Models Methods Appl. Sci.*, 19(7):1043–1064, 2009.

[10] P. Ciarlet, Jr., S. Sauter, and C. Simian. Non-Conforming Intrinsic Finite Elements for Elliptic Boundary Value Problems in Three Dimensions. Technical Report in preparation, 2015.

[11] M. Crouzeix and R. Falk. Nonconforming finite elements for Stokes problems. *Math. Comp.*, 186:437–456, 1989.

[12] M. Crouzeix and P. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *Revue Française d'Automatique, Informatique et Recherche Opérationnelle*, 3:33–75, 1973.

[13] M. Fortin and M. Soulie. A nonconforming quadratic finite element on triangles. *International Journal for Numerical Methods in Engineering*, 19:505–520, 1983.

[14] H. Lee and D. Sheen. Basis for the quadratic nonconforming triangular element of Fortin and Soulie. *International Journal of Numerical Analysis and Modeling*, 2(4):409–421, 2005.

[15] S. Mardare. On Poincaré and de Rham's theorems. *Revue Roumaine de Mathématiques Pures et Appliquées*, 53(5-6):523–541, 2008.

[16] J.-C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 35(3):315–341, 1980.

[17] J.-C. Nédélec. A new family of mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 50(1):57–81, 1986.

[18] S. Sauter and C. Schwab. *Boundary Element Methods*. Springer, Heidelberg, 2010.

[19] C. Schwab. *p- and hp-Finite Element Methods*. The Clarendon Press Oxford University Press, New York, 1998. Theory and applications in solid and fluid mechanics.

[20] C. Simian. *Intrinsic Discretization of Linear Elasticity*. PhD thesis, University of Zürich, 2013.

[21] G. Stoyan and Á. Baran. Crouzeix-Velte decompositions for higher-order finite elements. *Comput. Math. Appl.*, 51(6-7):967–986, 2006.